CHAPTER 16

COMPACTNESS

● PROBLEM 16-1

Why is any compact space also a Lindelöf space?

SOLUTION:

DEFINITION OF COMPACT SPACE

A space (X, T) is said to be compact if given any open cover $\{U_\omega\}$, $\omega \in \Omega$, of X, there is a finite subcover of $\{U_\omega\}$, $\omega \in \Omega$.

■

Let (X, T) represent any space and $A \subset X$. An open cover of A is a collection $\{U_\omega\}$, $\omega \in \Omega$ of open sets, such that

$$A \subset \bigcup_\omega U_\omega .$$

Also, $\{U_\omega\}$, $\omega \in \Omega$ is an open cover of A if $\{U_\omega \cap A\}$, $\omega \in \Omega$ is an open cover of the subspace A. A is said to be compact if every open cover of A has a finite subcover. Hence, A is compact if the subspace A is compact.

A space (X, T) is said to be a Lindelöf space if every open cover of X has a countable open subcover.

Therefore, any compact space is also Lindelöf.

● PROBLEM 16-2

Show that (0, 1) is not a compact space with the absolute value topology.

SOLUTION:

The open interval with the absolute value topology is Lindelöf since it is a subspace of a second countable space R. With the absolute value topology, (0, 1) is an example of a space which is Lindelöf, but not compact.

We shall now show that (0, 1) is not compact with the absolute value topology.

Consider the collection of open subsets of (0, 1)

$$\{U_n\},\ n = 1, 2, 3, 4, \ldots$$

where

$$U_n = \left(\frac{1}{n+1}, 1\right).$$

This is an open cover of (0, 1) because

$$(0,1) = \bigcup_n \left(\frac{1}{n+1}, 1\right).$$

No finite number of the U_n will cover (0, 1). Suppose, on the contrary that

$$U_{a_1}, U_{a_2}, \ldots, U_{a_k}$$

cover (0, 1). Let U_{a_l} represent the set with the highest index, then

$$(0,1) = U_{a_1} \cup U_{a_2} \cup \ldots \cup U_{a_k} = U_{a_l} = \left(\frac{1}{a_l + 1}, 1\right) \neq (0,1).$$

Hence, a contradiction.

We conclude that (0, 1) is not a compact space with the absolute value topology.

● PROBLEM 16–3

1. Use the Heine-Borel theorem to show that every closed and bounded interval $[a, b]$ with the absolute value topology is compact.

2. Show that every finite subset of a topological space (X, T) is compact.

SOLUTION:

1. We quote:

HEINE-BOREL THEOREM

Let $[a, b]$ represent a closed and bounded interval and let $\{U_\alpha\}$ represent a class of open sets, such that

$$[a,b] \subset \bigcup_\alpha U_\alpha.$$

Then one can select a finite number of the open sets, say

$$U_{\alpha_1}, U_{\alpha_2}, \ldots, U_{\alpha_n}$$

in such a way that

$$[a,b] \subset U_{\alpha_1} \cup U_{\alpha_2} \cup \ldots \cup U_{\alpha_n}.$$

■

According to the Heine-Borel theorem, every closed and bounded interval $[a, b]$ on the real line with the absolute value topology is compact.

2. Let $A = \{a_1, a_2, \ldots, a_n\}$ be a finite subset of a topological space (X, T). Let $\{U_\alpha\}$ be an open cover of A

$$A \subset \bigcup_\alpha U_\alpha .$$

Then each element of A belongs to at least one of the sets $\{U_\alpha\}$. Hence

$$a_1 \in U_{\alpha_1}$$

$$a_2 \in U_{\alpha_2}$$

$$\vdots$$

$$a_n \in U_{\alpha_n}$$

and

$$A \subset U_{\alpha_a} \cup \ldots \cup U_{\alpha_n}.$$

Any finite subset of a topological space is compact.

● PROBLEM 16–4

Let N represent the set of natural numbers with topology T defined as follows: a subset $U \subset N$ is open if it contains all, but mostly finite elements of N. Show that (N, T) is a compact space.

SOLUTION:

Let $\{U_\omega\}$, $\omega \in \Omega$ represent any open cover of N. We choose any element of $\{U_\omega\}$, say U_{ω_1}.

By definition U_{ω_1} contains all but most finitely elements of N. Let us denote these elements by

$$k_2, k_3, \ldots, k_m.$$

Since $\{U_\omega\}$ is an open cover of N, an open set U_{ω_2} exists, such that

$$k_2 \in U_{\omega_2}.$$

Similarly, we can find $U_{\omega_3}, \ldots, U_{\omega_m}$ such that

$$k_3 \in U_{\omega_3}, \ldots, k_m \in U_{\omega_m}.$$

For an open cover $\{U_\omega\}$, $\omega \in \Omega$ we found a finite subcover

$$\{U_{\omega_1}, U_{\omega_2}, \ldots, U_{\omega_m}\} \text{ of } \{U_\omega\}.$$

Since $\{U_\omega\}$, $\omega \in \Omega$ was an arbitrary cover of N, the space (N, T) is compact.

● PROBLEM 16-5

1. Show that any infinite subset A of a discrete topological space X is not compact.

2. Prove that a subset of a discrete space is compact, if and only if it is finite.

SOLUTION:

1. Let (X, T) represent a discrete topological space and A represent an infinite subset of A. We shall find an open cover of A which has no finite subcover. Consider the class of singleton sets

$$\{\{a\} : a \in A\}.$$

It is an open cover of A because

$$A = \bigcup_{a \in A} \{a\}$$

and since all subsets of a discrete space are open, each set $\{a\}$ is open. Note that no proper subclass of $\{\{a\} : a \in A\}$ is a cover of A.

Since A is infinite so is $\{\{a\} : a \in A\}$. Thus, the open cover $\{\{a\} : a \in A\}$ of A contains no finite subcover and A is not compact.

2. In Problem 16–3, we proved that every finite subset of a topological space is compact. Hence, a subset of a discrete space is compact, if and only if it is finite.

● PROBLEM 16-6

Show that a continuous image of a compact set is also compact.

SOLUTION:

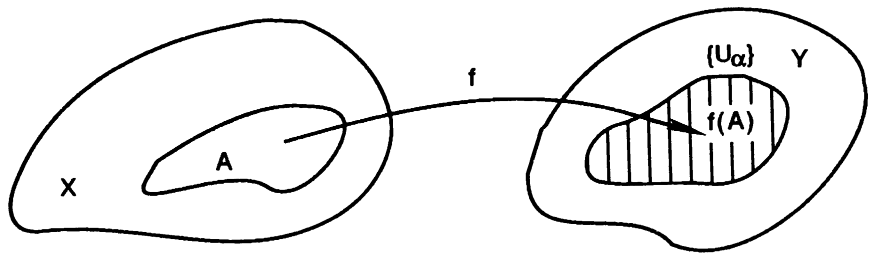


FIGURE 1

Suppose $f: X \to Y$ is continuous and X and Y are topological spaces and A is a compact subset of X.

Let $\{U_\alpha\}$ represent an open cover of $f(A)$

$$f(A) \subset \bigcup_\alpha U_\alpha .$$

Then, we have

$$A \subset f^{-1}[f(A)] \subset f^{-1}[\bigcup_\alpha U_\alpha] = \bigcup_\alpha f^{-1}(U_\alpha).$$

Since f is continuous, all sets $f^{-1}(U_\alpha)$ are open and $\{f^{-1}(U_\alpha)\}$ is an open cover of A. By hypothesis, A is a compact set, hence $\{f^{-1}(U_\alpha)\}$ is reducible to a finite cover, say

$$f^{-1}(U_{\alpha_1}),\ f^{-1}(U_{\alpha_2}), \dots ,\ f^{-1}(U_{\alpha_n}).$$

Then

$$A \subset f^{-1}(U_{\alpha_1}) \cup \dots \cup\ f^{-1}(U_{\alpha_n})$$

and

$$f(A) \subset f[f^{-1}(U_{\alpha_1}) \cup \dots \cup f^{-1}(U_{\alpha_n})] \subset U_{\alpha_1} \cup \dots \cup U_{\alpha_n}.$$

Thus $f(A)$ is compact.

● PROBLEM 16-7

Show that topological space (X, T) with the cofinite topology T is compact.

SOLUTION:

Suppose $\{U_\alpha\}$ is an open cover of X. Choose any

$$U \in \{U_\alpha\}.$$

Since T is the cofinite topology, $X - U$ is a finite set, which we can denote as

$$X - U = \{u_1, u_2, \dots, u_n\}.$$

$\{U_\alpha\}$ is a cover of X, therefore, for each of the elements of $\{u_1, u_2, \dots, u_n\}$, at least one set U_{α_i} exists, such that

$$u \in_i U_{\alpha_i} \text{ and } U_{\alpha_i} \in \{U_\alpha\}$$

for $i = 1, 2, \dots, n$.

Thus

$$X - U \subset U_{\alpha_1} \cup \dots \cup U_{\alpha_n}$$

and

$$X = U \cup (X - U) = U \cup U_{\alpha_1} \cup \ldots \cup U_{\alpha_n}.$$

Hence X is compact.

• PROBLEM 16-8

1. Find an example of a subset of a compact space which is not compact.

2. Prove the following

THEOREM

A closed subset of a compact space is also compact.

■

SOLUTION:

1. Consider the closed interval $[a, b]$ which is compact by the Heine-Borel theorem. Open interval (a, b), which is a subset of $[a, b]$, is not compact (see Problem 16–2).

2. Let $F \subset X$ represent a closed subset of a compact space (X, T) and let

$$\{U_\alpha\}$$

represent an open cover of F,

$$F \subset \bigcup_\alpha U_\alpha.$$

We have

$$X = (X - F) \cup \bigcup_\alpha U_\alpha.$$

Since F is closed, $X - F$ is open and

$$\{U_\alpha, X - F\}$$

is an open cover of X.

By hypothesis, X is compact and the cover $\{U_\alpha, X - F\}$ is reducible to a finite cover, say

$$X = U_1 \cup \ldots \cup U_n \cup (X - F)$$

where $U_1, \ldots, U_n \in \{U_\alpha\}$.

Since F and $X - F$ are disjoint we obtain

$$F \subset U_1 \cup \ldots \cup U_n.$$

Therefore, an open cover $\{U_\alpha\}$ of F is reducible to a finite subcover $\{U_1, U_2, \ldots, U_n\}$.

Set F is compact.

● PROBLEM 16-9

Show that the following conditions are equivalent:

1. X is compact
2. For every family $\{F_\alpha\}$ of closed subsets of X,

$$\left(\bigcap_\alpha F_\alpha = \phi\right) \Rightarrow \begin{pmatrix} \{F_\alpha\} \text{ contains a finite subclass} \\ \{F_{\alpha_1}, \ldots, F_{\alpha_k}\} \text{ such that} \\ F_{\alpha_1} \cap \ldots \cap F_{\alpha_k} = \phi \end{pmatrix}.$$

SOLUTION:

1. $\Rightarrow$ 2.

Suppose $\bigcap_\alpha F_\alpha = \phi$, then according to de Morgan's law

$$X = X - \left(\bigcap_\alpha F_\alpha\right) = \bigcup_\alpha (X - F_\alpha)$$

$\{X - F_\alpha\}$ is an open cover of X, because all F_α are closed. But, by hypothesis X is compact, hence a finite subcover exists, say

$$X = (X - F_{\alpha_1}) \cup \ldots \cup (X - F_{\alpha_k})$$

where

$$F_{\alpha_1}, \ldots, F_{\alpha_k} \in \{F_\alpha\}.$$

Again, according to de Morgan's law

$$\phi = X - X = X - [(X - F_{\alpha_a}) \cup \ldots \cup (X - F_{\alpha_k})] = F_{\alpha_1} \cap \ldots \cap F_{\alpha_n}.$$

2. $\Rightarrow$ 1.

Let $\{U_\alpha\}$ represent an open cover of X.

$$X = \bigcup_\alpha U_\alpha.$$

According to de Morgan's law

$$\phi = X - X = \bigcap_{\alpha} (X - U_{\alpha}).$$

All sets $X - U_\alpha$ are closed and have an empty intersection. By hypothesis, a subclass of $\{X - U_\alpha\}$ exists, say $(X - U_{\alpha_1})$, …, $(X - U_{\alpha_k})$, such that

$$(X - U_{\alpha_1}) \cap \ldots \cap (X - U_{\alpha_k}) = \phi.$$

According to de Morgan's law

$$X - [(X - U_{\alpha_1}) \cap \ldots \cap (X - U_{\alpha_k})] = U_{\alpha_1} \cup U_{\alpha_2} \cup \ldots \cup U_{\alpha_k} = X.$$

Thus, X is compact.

● PROBLEM 16–10

> Show that the family of open intervals
>
> $$\{(0, 1), (0, {}^1/_2), (0, {}^1/_3), (0, {}^1/_4), \ldots\}$$
>
> has the finite intersection property.

SOLUTION:

DEFINITION

A family $\{A_\alpha\}$ of sets is said to have the finite intersection property, if every finite subfamily

$$\{A_{\alpha_1}, \ldots, A_{\alpha_n}\}$$

has a non-empty intersection, that is

$$A_{\alpha_1} \cap \ldots \cap A_{\alpha_n} = \phi.$$

■

Let $\{(0, a_1), \ldots, (0, a_n)\}$ represent any subfamily of open intervals, then

$$(0, a_1) \cap \ldots \cap (0, a_n) = (0, a) \neq \phi$$

where

$$a = min\,(a_1, a_2, \ldots, a_n).$$

Observe that the intersection of all the members of the family

$$(0, 1) \cap (0, {}^1/_2) \cap \ldots = \phi$$

is empty.

• PROBLEM 16–11

Prove this theorem:

THEOREM

A topological space (X, T) is compact, if and only if every class $\{F\alpha\}$ of closed subsets of X, which has the finite intersection property (see Problem 16–10), has a non-empty intersection, i.e.,

$$\bigcap_{\alpha} F_{\alpha} \neq \phi.$$

■

SOLUTION:

The implication of Problem 16–9 can be written in the form

$$\left(\bigcap_{\alpha} F_{\alpha} \neq \phi\right) \Rightarrow \begin{pmatrix} \exists\ \alpha_1, \alpha_2, \dots, \alpha_k \text{ such that} \\ F_{\alpha_1} \cap F_{\alpha_2} \cap \dots \cap F_{\alpha_k} = \phi \end{pmatrix} \tag{1}$$

Let a and b represent sentences, then

$$(a \Rightarrow b) \equiv (a' \vee b)$$

and

$$(b' \Rightarrow a') \equiv (b \vee a').$$

Let us take the negation of (1)

$$\begin{pmatrix} \forall\ \alpha_1, \alpha_2, \dots, \alpha_k \text{ such that} \\ F_{\alpha_1} \cap F_{\alpha_2} \cap \dots \cap F_{\alpha_k} = \phi \end{pmatrix} \Rightarrow \left(\bigcap_{\alpha} F_{\alpha} \neq \phi\right). \tag{2}$$

Since (1) implies (2), the proof is completed.

• PROBLEM 16–12

Let A represent a compact subset of a Hausdorff space (X, T). Show that, if

$$q \in X - A, \tag{1}$$

then the open sets U and V exist, such that

$$q \in U, \quad A \subset V, \quad U \cap V = \phi. \tag{2}$$

See Figure 1.

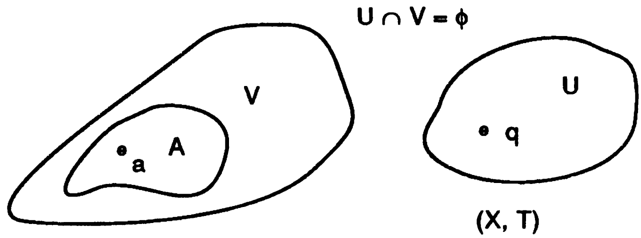


FIGURE 1

SOLUTION:

Choose $a \in A$, since $q \in X - A$, $a \neq q$ and since (X, T) is a Hausdorff space, open sets U_a and V_a exist such that

$$q \in U_a, \quad a \in V_a, \quad U_a \cap V_a = \phi. \tag{3}$$

The family of sets $\{V_a : a \in A\}$ forms an open cover of A

$$A \subset \bigcup_{a \in A} V_a. \tag{4}$$

Since A is compact, a finite subcover $V_{a_1}, V_{a_2}, \dots, V_{a_k}$ exists, such that

$$A \subset V_{a_1} \cup V_{a_2} \cup \dots \cup V_{a_k}. \tag{5}$$

Let us define

$$V = V_{a_1} \cup V_{a_2} \cup \dots \cup V_{a_k} \tag{6}$$

and

$$U = U_{a_1} \cap \dots \cap U_{a_k}. \tag{7}$$

Obviously U and V are open as the union and finite intersection of open sets.

From (5) and (6) we have

$$A \subset V,$$

from (3) and (7) we have

$$q \in U.$$

We have to show that $U \cap V = \phi$.

$$U \cap V = (U_{a_1} \cap \dots \cap U_{a_k}) \cap (V_{a_1} \cup V_{a_2} \cup \dots \cup V_{a_k}). \tag{8}$$

Since

$$U_{a_l} \cap V_{a_l} = \phi \tag{9}$$

for $l = 1, 2, \dots, k$, we have

$$U \cap V = \phi.$$

● **PROBLEM 16-13**

Show that any compact subset of a Hausdorff space is closed.

SOLUTION:

Let A represent a compact subset of a Hausdorff space (X, T) and let $q \notin A$.

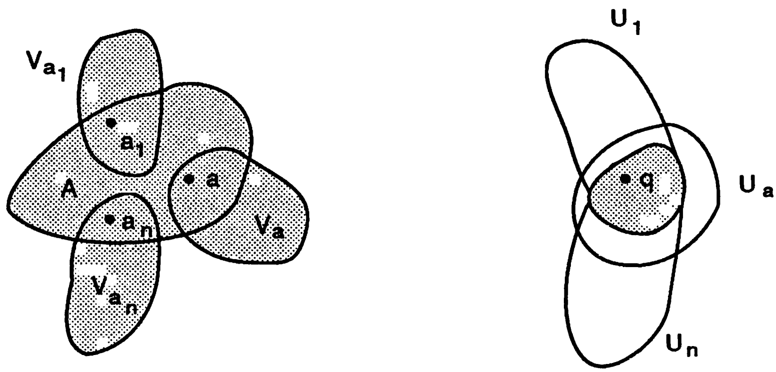


FIGURE 1

We shall show that if A is compact, then $X - A$ is open.

Let $a \in A$; then since (X, T) is Hausdorff, there are neighborhoods U_a and V_a of a and q respectively, such that

$$a \in U_a, \quad q \in V_a, \quad U_a \cap V_a = \phi. \tag{1}$$

The family of sets

$$\{V_a : a \in A\} \tag{2}$$

forms an open cover of A. Since A is compact, there are finitely many

$$a_1, a_2, \ldots, a_n \text{ such that } \{V_{a_1}, \ldots, V_{a_n}\} \tag{3}$$

forms on an open cover of A. Let

$$U = U_{q_1} \cap \ldots \cap U_{q_n} \text{ and } V = V_{a_1} \cup \ldots \cup V_{a_n}. \tag{4}$$

Then $q \in U$ and $U \cap V = \phi$. Since $q \in X - A$ and $q \in U$ and $U \cap A = \phi$, then $X - A$ is open and A is closed.

● **PROBLEM 16-14**

Show that a subset of a compact T_2–space (Hausdorff space) is compact, if and only if it is closed.

SOLUTION:

Let (X, T) represent a Hausdorff compact space and $A \subset X$, then

$$(A \text{ is compact}) \Leftrightarrow (A \text{ is closed}).$$

$\Leftarrow$ In Problem 16–8 we proved that any closed subset of a compact space is compact.

$\Rightarrow$ In Problem 16–13 we proved that any compact subset of a Hausdorff space is closed.

● PROBLEM 16–15

Prove

THEOREM

Let A and B represent disjoint compact subsets of a Hausdorff space (X, T). Then open sets U and V exist, such that

$$A \subset U, \quad B \subset V, \quad U \cap V = \phi. \tag{1}$$

■

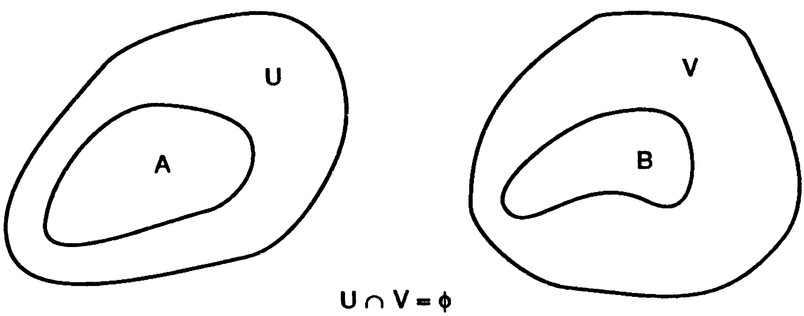


FIGURE 1

SOLUTION:

Let $a \in A$: then, since $A \cap B = \phi$, $a \notin B$. By hypothesis, B is compact and open sets U_a and V_a exist, such that

$$a \in U_a, \quad B \subset V_a, \quad U_a \cap V_a = \phi. \tag{2}$$

The family of sets $\{U_a : a \in A\}$ forms an open cover of A and since A is compact we can select $\{U_{a_1}, U_{a_2}, \dots, U_{a_n}\}$ a finite subcover of A.

Let us denote

$$U = U_{a_1} \cup \ldots \cup U_{a_n}$$

$$V = V_{a_1} \cap \ldots \cap V_{a_n}. \tag{3}$$

Then

$$A \subset U \quad \text{and} \quad B \subset V. \tag{4}$$

Both U and V are open sets. For each $k = 1, 2, \ldots, n$

$$U_{a_k} \cap V_{a_k} = \phi. \tag{5}$$

Thus

$$U_{a_k} \cap V = \phi$$

and

$$U \cap V = (U_{a_1} \cup \ldots \cup U_{a_n}) \cap V = \phi. \tag{6}$$

● PROBLEM 16–16

Show that every compact Hausdorff space is normal.

SOLUTION:

Suppose (X, T) is a compact Hausdorff space and F_1 and F_2 are disjoint closed subsets of X.

By Problem 16–8, both sets F_1 and F_2 are also compact.

By Problem 16–15, open subsets U_1 and U_2 exist such that

$$F_1 \subset U_1, \quad F_2 \subset U_2, \quad U_1 \cap U_2 = \phi.$$

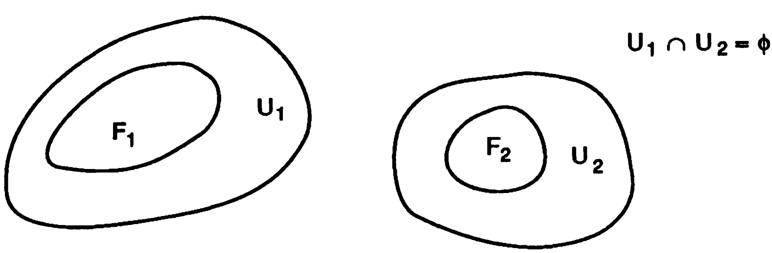


FIGURE 1

We conclude that (X, T) is normal (see definition in Problem 13–22). See Figure 2.

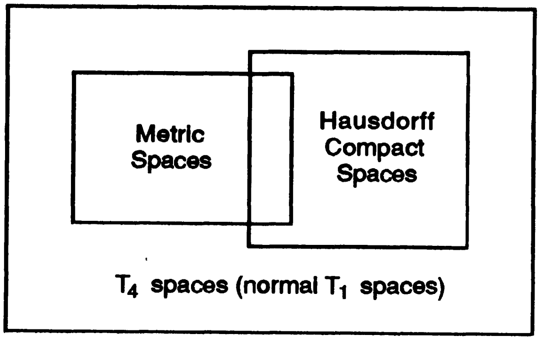


FIGURE 2

● PROBLEM 16–17

1. Let A represent a compact subset of a Hausdorff space (X, T).

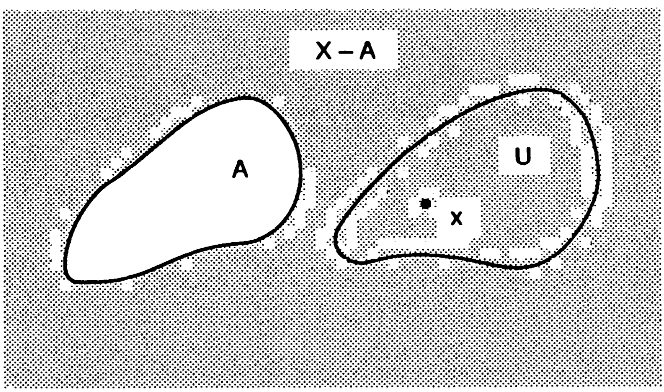


FIGURE 1

Show that if $x \notin A$, then there is an open set U, such that

$$x \in U \subset X - A. \tag{1}$$

2. Prove that a compact subset A of a Hausdorff space (X, T) is closed.

SOLUTION:

1. Open sets U and V exist such that

$$x \in U, \quad A \subset V \quad \text{and} \quad U \cap V = \phi. \tag{2}$$

Therefore,

$$U \cap A = \phi \quad \text{and} \quad x \in U \subset X - A. \tag{3}$$

2. We shall prove that $X - A$ is open. Let

$$x \in X - A$$

that is,

$$x \notin A.$$

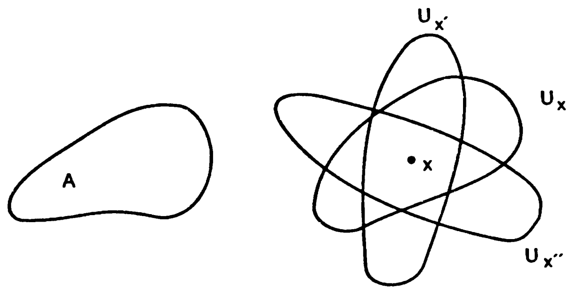


FIGURE 2

Since A is compact, an open set U_x exists, such that

$$x \in U_x \subset X - A. \tag{4}$$

Therefore,

$$X - A = \bigcup_{x \in X - A} U_x \tag{5}$$

and $X - A$ is open, while the union of open sets and A is closed. Compare with Problem 16–13.

● PROBLEM 16–18

How does an equivalence relation lead to the new compact spaces? (Hint: use results of Problem 16–17).

Show that the circle is compact.

SOLUTION:

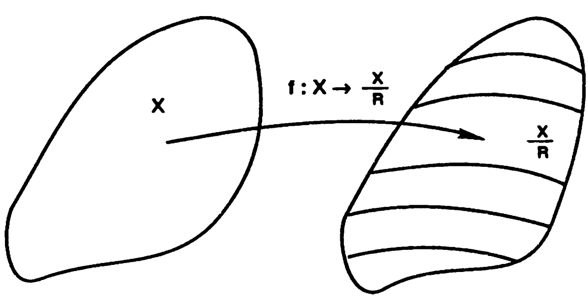


FIGURE 1

Suppose X is a compact space and R is an equivalence relation on X.

The space ${}^{X}/_{R}$ is called the identification space. Since the identification mapping f from X onto ${}^{X}/_{R}$,

$$f : X \rightarrow {}^{X}/_{R}$$

is continuous and the space ${}^{X}/_{R}$ is compact.

We have already shown that the closed interval [0, 1] is compact. An equivalence relation between [0, 1] and the circle can be established as follows:

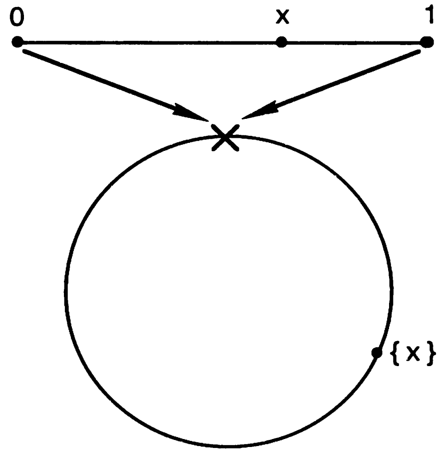


FIGURE 2

Let 0 be equivalent to 1, and every other element of [0, 1] be equivalent only to itself.

By identifying 0 and 1, we obtain a circle. The circle is an identification space derived from [0, 1], hence the circle is compact.

● PROBLEM 16–19

Prove:

THEOREM

Let f represent a one-to-one continuous function

$$f : X \rightarrow Y$$

from a compact space X into a Hausdorff space Y. Then X and $f(X)$ are homeomorphic.

■

See Figure 1.

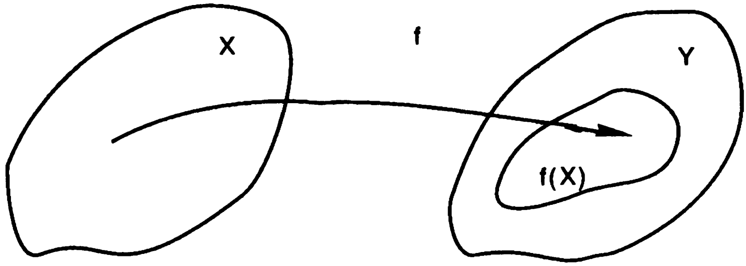


FIGURE 1

SOLUTION:

Function

$$f : X \to f(X) \tag{1}$$

is a continuous bijection (i.e., one-to-one and onto). Hence,

$$f^{-1} : f(X) \to X \tag{2}$$

exists. By hypothesis, f is continuous, so we only have to show that f^{-1} is continuous.

Function f^{-1} is continuous if, for every closed subset F of X,

$$[f^{-1}]^{-1}(F) = f(F) \tag{3}$$

it is a closed subset of $f(X)$.

Let F represent a closed subset of a compact space, then (by Problem 16–8) F is compact. Since F is continuous, $f(F)$ is a compact subset of $f(X)$. $f(X)$ is the Hausdorff space as a subspace of the Hausdorff space Y. Thus, $f(F)$ is closed and f^{-1} is continuous. Therefore,

$$f : X \to f(X) \tag{4}$$

is a homeomorphism.

● PROBLEM 16–20

Let (X, T) represent compact and let (X, T') represent the Hausdorff space.

Prove that

$$(T' \subset T) \Rightarrow (T' = T).$$

SOLUTION:

Consider the identity function defined on X:

$$f(x) = x$$

$$f: (X, T) \to (X, T').$$

Function f is one-to-one and onto. Since

$$T' \subset T,$$

f is continuous.

Since (X, T) is compact and (X, T') is the Hausdorff space, we conclude that (X, T) and (X, T') are homeomorphic. Therefore

$$T' = T.$$

● PROBLEM 16–21

Here is an important theorem which enables us to find more compact spaces.

TYCHONOFF THEOREM

Let $\underset{k}{\times} X_k$ $k \in K$ represent the product space of the countable family of non-empty spaces $\{(X_K, T_k) : k \in K\}$. Then

$$\underset{k}{\times} X_k$$

is compact, if and only if each component space is compact.

■

Prove it.

SOLUTION:

Suppose $\underset{k}{\times} X_k$ is compact. For each $l \in K$, the projection map

$$p_l : \underset{k}{\times} X_k \to X_l$$

is continuous and onto.

Therefore X_l is compact for each $l \in K$. Now, suppose each X_k is compact. Let $\{t_j\}, j \in J$ represent any ultranet in $\underset{k}{\times} X_k$ with the kth coordinate of t_j denoted by $t_j(k)$. Then

$$\{p_k(t_j)\} = \{t_j(k)\}, j \in J$$

is an ultranet in X_k and $\{t_j(k)\}$ converges in X_k. Therefore, $\{t_k\}$, $k \in K$ converges in $\underset{k}{\times} X_k$.

A space (X, T) is compact, if and only if every ultranet in X converges. Hence,

$$\underset{k}{\times} X_k$$

is compact.

● PROBLEM 16–22

Show that the cylinder, the torus and the cube are compact spaces.

SOLUTION:

We have already proved that the closed interval $I = [0, 1]$ and the circle C are compact.

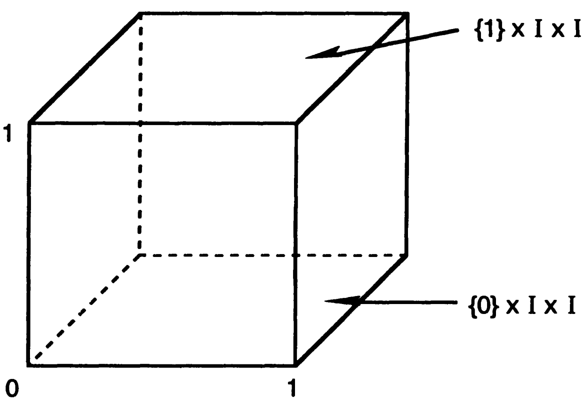


FIGURE 1

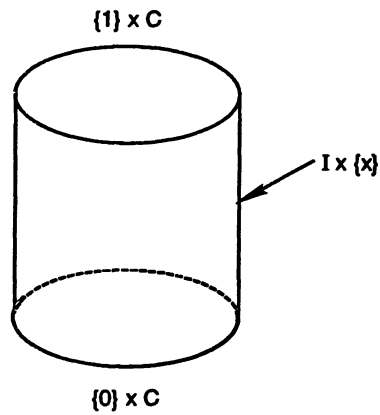


FIGURE 2

According to the Tychonoff theorem, the cube $[0, 1] \times [0, 1] \times [0, 1]$ is compact because $[0, 1]$ is compact.

Similarly, we show that the cylinder is compact, see Figure 2.

Since C and $[0, 1]$ are compact, the cylinder $C \times [0, 1]$ is compact.

The torus $C \times C$ is also compact, see Figure 3.

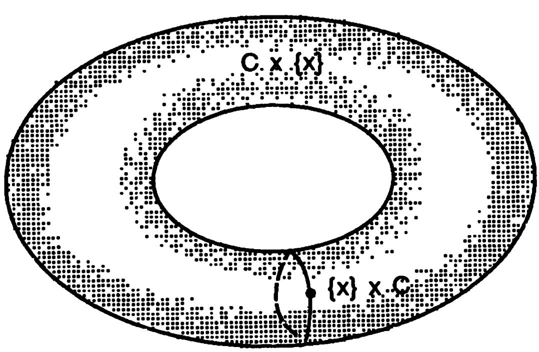


FIGURE 3

1. Show that if A is compact and F is closed, then $A \cap F$ is compact.

2. Prove that if $A_1, A_2, \ldots, A_k$ are compact subsets of a topological space X, then $A_1 \cup \ldots \cup A_k$ is also compact.

SOLUTION:

1.

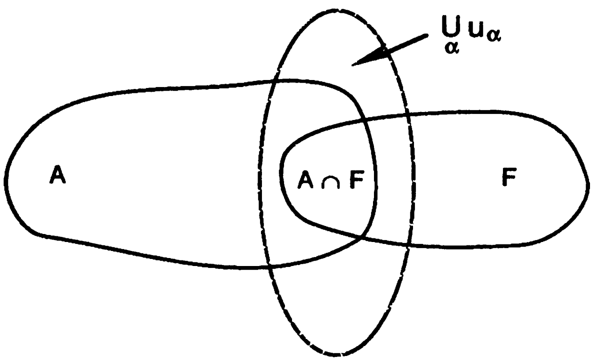


FIGURE 1

Let $\{U_\alpha\}$ represent an open cover of $A \cap F$

$$A \cap F \subset \bigcup_\alpha U_\alpha. \tag{1}$$

Then

$$A \subset \bigcup_\alpha U_\alpha \cup (X - F). \tag{2}$$

and since F is closed $\{\{U_\alpha\}, X - F\}$ is an open cover of A. Set A is compact so we can choose a finite subcover, say

$$A \subset U_{\alpha_1} \cup \ldots \cup U_{\alpha_n} \cup (X - F). \tag{3}$$

Hence

$$A \cap F \subset U_{\alpha_1} \cup \ldots \cup U_{\alpha_n} \tag{4}$$

and $A \cap F$ is compact.

2. Let A and B represent compact subsets of (X, T) and let $\{U_\alpha\}$ represent an open cover of $A \cup B$

$$A \cup B \subset \bigcup_\alpha U_\alpha. \tag{5}$$

Then $\{U_\alpha\}$ is an open cover of A and B, which are compact.

$$A \subset \bigcup_\alpha U_\alpha. \tag{6}$$

We can choose finite subcover

$$A \subset U_{\alpha_1} \cup \ldots \cup U_{\alpha_n} \tag{7}$$

and

$$B \subset U_{\alpha_{n+1}} \cup \ldots \cup U_{\alpha_{n+k}}. \tag{8}$$

Hence

$$A \cup B \subset U_{\alpha_1} \cup \ldots \cup U_{\alpha_{n+k}} \tag{9}$$

and $A \cup B$ is compact.

● PROBLEM 16–24

Show that a finite subset A of a topological space (X, T) is sequentially compact.

SOLUTION:

DEFINITION OF SEQUENTIALLY COMPACT SETS

A subset A of a topological space (X, T) is sequentially compact if every sequence in A contains a subsequence which converges to a point in A. ■

Let A represent a finite subset of (X, T), and let

$$(x_1, x_2, x_3, \ldots) \tag{1}$$

represent a sequence in A.

Since A is finite, at least one of the elements in A, say x_0, appears an infinite number of times in the sequence. We can choose the subsequence of (1) to be

$$(x_0, x_0, x_0, \ldots) \tag{2}$$

which converges to the point x_0 belonging to A.

● PROBLEM 16–25

Show that a continuous image of a sequentially compact set is sequentially compact.

SOLUTION:

Let

$$f : X \to Y \tag{1}$$

be a continuous function and let A represent a subset of X which is sequentially compact. We will show that $f(A)$ is sequentially compact in Y.

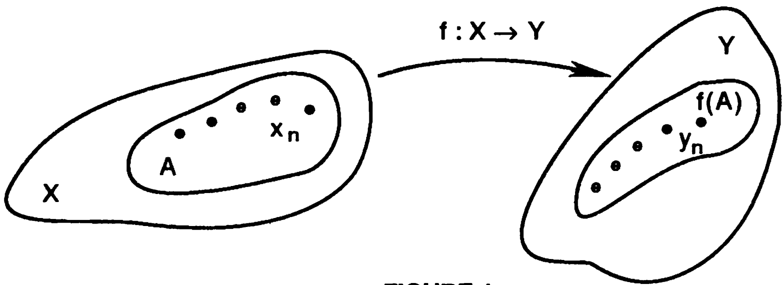


FIGURE 1

Let

$$(y_1, y_2, \dots) \tag{1}$$

represent a sequence in $f(A)$. Then there exists a sequence in A

$$(x_1, x_2, \dots) \tag{2}$$

such that

$$f(x_1) = y_1, f(x_2) = y_2, \dots, f(x_n) = y_n, \dots.$$

Since A is sequentially compact, sequence (2) contains a subsequence, say

$$(x_{k_1}, x_{k_2}, \dots) \tag{3}$$

which converges to a point $x_0 \in A$.

Since f is continuous, we have

$$\big((x_{k_1}, x_{k_2}, \dots) \text{ converges to } x_0\big) \Rightarrow \begin{pmatrix} (f(x_{k_1}), f(x_{k_2}), \dots) \\ \text{converges to } f(x_0) \end{pmatrix}.$$

We have $f(x_0) \in f(A)$.

Thus, $f(A)$ is sequentially compact.

● PROBLEM 16–26

1. Show that every bounded closed interval $[a, b]$ is countably compact.
2. Show that the open interval (a, b) is not countably compact.

SOLUTION:

DEFINITION OF COUNTABLY COMPACT SETS

A subset A of a topological space (X, T) is countably compact, if every infinite subset $B \subset A$ has an accumulation point in A.

■

1. Let B represent an infinite subset of a closed interval $[a, b]$.

$$B \subset [a, b].$$

We shall apply:

BOLZANO-WEIERSTRASS THEOREM

Every bounded infinite set of real numbers has an accumulation point.

■

Thus B has an accumulation point x. Since $[a, b]$ is closed, and $B \subset [a, b]$, the accumulation point x of B belongs to $[a, b]$. Hence $[a, b]$ is countably compact.

2. We shall show that the open interval $(0, 1)$ is not countably compact. Consider the infinite subset of $(0, 1)$

$$\{1/2, 1/3, 1/4, 1/5, \ldots\}.$$

This subset has only one accumulation point 0 which does not belong to $(0, 1)$.

Hence $(0, 1)$ is not countably compact.

● PROBLEM 16–27

Show that a closed subset of a countably compact space is countably compact.

SOLUTION:

Let (X, T) represent a countably compact space and F be a closed subset of X.

Let A represent any infinite subset of F. A is also an infinite subset of a countably compact space (X, T). Therefore an accumulation point x of A exists, such that

$$x \in X$$

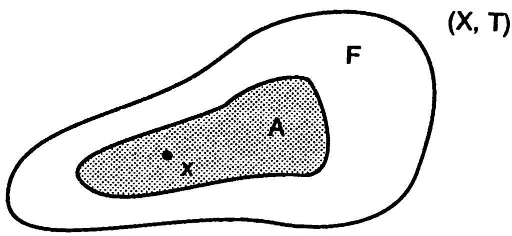


FIGURE 1

Since $A \subset F$, x is also an accumulation point of F. By hypothesis, F is closed, so it contains all its accumulation points. Thus

$$x \in F.$$

Hence, any infinite subset A of a closed set F has an accumulation point $x \in F$. Therefore, F is countably compact.

● PROBLEM 16–28

Prove:

$$(X \text{ is compact}) \Rightarrow (X \text{ is countably compact}).$$

SOLUTION:

Suppose (X, T) is compact.
Let A represent a subset of X

$$A \subset X$$

with no accumulation points in X.

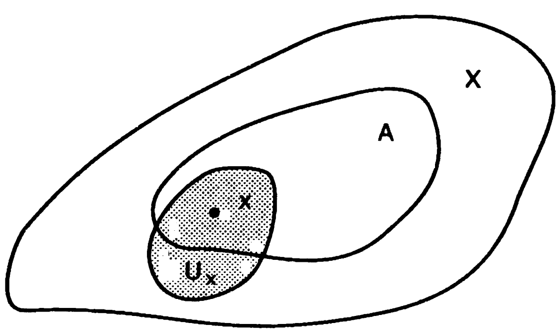


FIGURE 1

Each point $x \in X$ belongs to an open set U_x which contains, at most, one point of A.

Now, consider the family of sets

$$\{U_x : x \in X\} \tag{1}$$

Family (1) consists of open sets and

$$X \subset \bigcup_{x \in X} U_x. \tag{2}$$

Hence (1) is an open cover of X. Since X is compact (by hypothesis), we can select a finite subcover

$$U_{x_1}, U_{x_1}, \ldots, U_{x_n}. \tag{3}$$

Then

$$A \subset X \subset U_{x_1} \cup \ldots \cup U_{x_n}. \tag{4}$$

Since each of the sets (3) contains, at most, one point of A, A is finite. Therefore, every infinite subset of X contains an accumulation point in X, that is, X is countably compact.

● PROBLEM 16–29

In Problem 16–28 we showed that if the space is compact, then it is countably compact.

$$((X, T) \text{ compact}) \Rightarrow ((X, T) \text{ countably compact}).$$

Give an example showing that this implication cannot be reversed.

SOLUTION:

We shall give an example of a space which is countably compact, but not compact. Let N represent the set of positive integers with the topology T generated by the sets

$$\{1, 2\}, \{3, 4\}, \{5, 6\}, \ldots$$

and let $A \neq \phi$ represent a non-empty subset of N.

Let $k \in A$, then, if k is even, then $k - 1$ is a limit point of A and if k is odd, then $k + 1$ is a limit point of A.

Hence A has an accumulation point and (N, T) is countably compact. Since

$$\{\{1, 2\}, \{3, 4\}, \{5, 6\}, \ldots\}$$

is an open cover of N which has no finite subcover, (N, T) is not compact.

● PROBLEM 16–30

1. Show that sequential compactness is a topological property.

2. Show that countable compactness is a topological property.

SOLUTION:

1. Let (X, T) and (Y, T') represent homeomorphic topological spaces. Then a homeomorphism exists

$$f : X \rightarrow Y.$$

Suppose X is sequentially compact and let

$$(y_n) \tag{1}$$

be a sequence in Y.

Then, since f is onto and one-to-one,

$$f^{-1}(y_n) \tag{2}$$

is a sequence in X. Since X is sequentially compact, (2) contains a subsequence, say

$$f^{-1}(y_{k_n}) \tag{3}$$

which converges to a point in X,

$$f^{-1}(y_{k_n}) \rightarrow x \in X. \tag{4}$$

Since f is continuous,

$$f(f^{-1}(y_{k_n})) = (y_{k_n}) \rightarrow f(x) \in Y. \tag{5}$$

Thus, (1) contains a subsequence (y_{k_n}), which converges to a point $f(x)$ in Y. (Y, T') is sequentially compact.

2. Let B represent an infinite subset of Y. Then $f^{-1}(B)$ is an infinite subset of X. X is countably compact. Hence $f^{-1}(B)$ has an accumulation point x in X. Thus B has an accumulation point $f(x)$ in Y and Y is countably compact.

● PROBLEM 16–31

Prove that if a space (X, T) is sequentially compact, then it is countably compact.

$$(X \text{ sequentially compact}) \Rightarrow (X \text{ countably compact}).$$

SOLUTION:

Suppose (X, T) is sequentially compact and A is any infinite subset of $X, A \subset X$. We shall show that A has an accumulation point in A. We can find a sequence

$$(a_n) \quad n \in N$$

in A with distinct terms, i.e., for any $k, l \in N, k \neq l$

$$a_k \neq a_l.$$

Since X is sequentially compact, (a_n) contains a subsequence which converges to a point in X.

$$(a_{k_n}) \to x \in X.$$

Then $x \in X$ is an accumulation point of A and (X, T) is countably compact.

There are some other properties related to compactness, for example, paracompactness, metacompactness, and pseudocompactness.

● PROBLEM 16–32

Show that the coordinate plane R^2 with the Pythagorean metric topology is locally compact.

SOLUTION:

DEFINITION OF LOCALLY COMPACT SPACES

A space (X, T) is said to be locally compact if for any $x \in X$ and any neighborhood U of x, there is a compact set A such that

$$x \in Int(A) \subset A \subset U \tag{1}$$

■

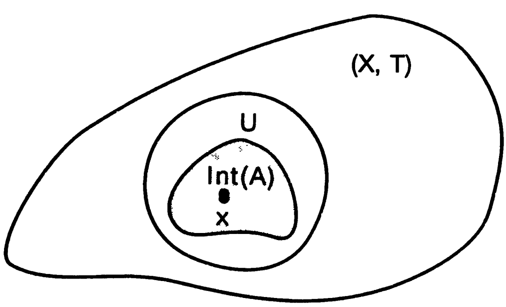


FIGURE 1

Plane R^2 is not compact, since it is not bounded.

Let

$$x \in R^2$$

and let U represent any neighborhood of x. Then there exists $r > 0$ such that

$$x \in B(x, r) \subset U. \tag{2}$$

We have

$$B(x, {}^r/_2) \subset \overline{B(x, {}^r/_2)} \subset B(x, r) \subset U. \tag{3}$$

The set

$$\overline{B(x, {}^r/_2)}$$

is a closed and bounded subset of R^2, hence it is compact.

$$x \in B(x, {}^r/_2) \subset \overline{B(x, {}^r/_2)} \subset U. \tag{4}$$

Hence R^2 with the Pythagorean metric topology is locally compact.

● PROBLEM 16–33

Prove that any compact T_2–space (X, T) is locally compact.

SOLUTION:

The criterion for local compactness given in Problem 16–32 is simpler for the T_2–spaces.

DEFINITION

Let (X, T) represent a T_2–space. Then X is locally compact if and only if, given any

$$x \in X,$$

there is a compact set A, such that

$$x \in Int(A).$$

■

One can prove that for T_2–spaces, the existence of one compact subset A of X such that $x \in Int(A)$ guarantees that for any neighborhood U of x, there is a compact set B, such that

$$x \in Int(B) \subset B \subset U.$$

Let (X, T) be a compact T_2–space. Then X is a compact neighborhood of any $x \in X$.

● PROBLEM 16–34

Let Q represent the subspace of rational numbers in the space R of real numbers with the absolute value topology. Show that Q is not locally compact.

SOLUTION:

Space R is obviously locally compact.

Take any $x \in Q$ and a compact set A such that

$$x \in int(A). \tag{1}$$

Remember that a subset A of R (or R^n) is compact, if and only if A is bounded and closed.

Then A contains infinitely many elements of Q.

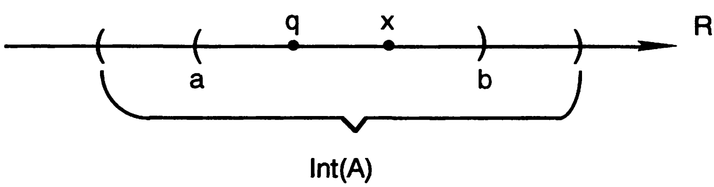


FIGURE 1

We can find rational numbers a and b, such that $(a, b) \subset R$

$$x \in (a, b) \cap Q \subset Int(A). \tag{2}$$

Let q represent an irrational number, such that

$$q \in (a, b). \tag{3}$$

We define an open cover of A as follows:

for each $a \in A$,

$$U(a) = \begin{cases} if \;\; q < a \;\; \{s \in R : a < s\} \\ if \;\; q > a \;\; \{t \in R : t < a\} \end{cases} \tag{4}$$

$\{U(a) \cap A : a \in A\}$ is an open cover of A which has not finite subcover. Hence Q is not locally compact.

• PROBLEM 16–35

Prove the following

THEOREM

If (X, T) is a Hausdorff space (i.e., a T_2–space) locally compact, then (X, T) is a regular space

■

SOLUTION:

We shall apply the results of Problem 13–17. A space (X, T) is regular iff given any $x \in X$ and any neighborhood U of x, there is a neighborhood V of x such that

$$\overline{V} \subset U.$$

Let (X, T) represent a locally compact T_2–space. If $x \in X$, and U is any neighborhood of x, then there is a compact set A, such that

$$x \in Int(A) \subset A \subset U \tag{1}$$

(see Problem 16–32).

Since A is compact, A is closed, hence

$$\overline{Int(A)} \subset A. \tag{2}$$

We define V as

$$V = Int(A) \tag{3}$$

and obtain

$$x \in V \subset \overline{V} \subset U. \tag{4}$$

Since V is a neighborhood of x, (X, T) is regular.

• PROBLEM 16–36

1. Show that an open subspace of a locally compact space is locally compact.

2. Show that a closed subspace of a locally compact T_2–space is locally compact.

SOLUTION:

1. Let U represent an open subspace of a locally compact space (X, T) and $x \in U$.

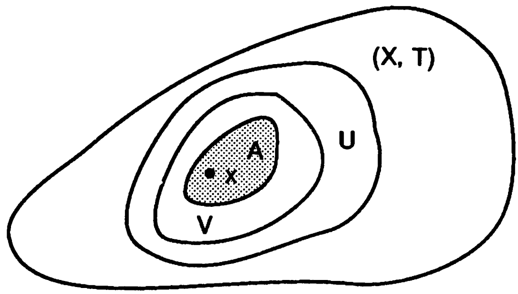


FIGURE 1

Let V represent any neighborhood of x, such that

$$x \in V \subset U. \tag{1}$$

Since (X, T) is locally compact, a compact set A exists such that

$$x \in Int(A) \subset A \subset V. \tag{2}$$

Hence, U is locally compact.

2. Let F represent a closed subspace of X and $x \in F$ and let A represent any compact subset of X, such that

$$x \in Int(A) \subset A. \tag{3}$$

A is closed because (X, T) is T_2. The set

$$F \cap A$$

is a closed subset of the compact set A and thus, is compact. We have

$$x \in Int_F(F \cap A) \subset F \cap A \subset F \tag{4}$$

Where $Int_F(F \cap A)$ denotes $Int(F \cap A)$ in F.

By definition in Problem 16–33, we conclude that since F is T_2, F is locally compact.

● PROBLEM 16–37

In Problem 16–6, we showed that compactness is preserved by continuous functions. One can find examples showing that local compactness is not preserved by continuous functions. Prove the following:

THEOREM

If $f: (X, T) \to (Y, T')$ is a continuous, open function from a locally compact space (X, T) onto Y, then (Y, T') is also locally compact. ∎

SOLUTION:

A function $f: X \to Y$ is said to be open, if for any open subset $U \subset X$, $f(U)$ is open in Y.

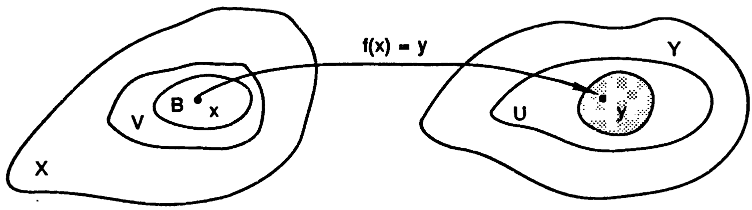


FIGURE 1

Let $y \in Y$ and U represent a neighborhood of y. For some $x \in X$

$$f(x) = y.$$

Since f is continuous, an open set V exists, such that

$$x \in V \in T \quad \text{and} \quad f(V) \subset U.$$

Since (X, T) is locally compact, there is a compact set B such that

$$x \in Int(B) \subset B \subset V.$$

Then

$$y = f(x) \in f(Int\, B) \subset f(B) \subset U.$$

Since f is open, $f(Int\, B)$ is open. Since B is compact and f is continuous, $f(B)$ is compact. Thus, Y is locally compact.

● PROBLEM 16–38

Prove the existence of a Lebesque number described in this theorem.

THEOREM

If (X, d) is a compact metric space and $\{U_\alpha\}$ is an open cover of X, then there is a positive number $r > 0$, such that

$$B(x, r) \subset U_\alpha \text{ for some } \alpha \text{ for any } x \in X.$$

∎

That is, number $r > 0$ exists such that the r–neighborhood $B(x, r)$ of any point $x \in X$ is a subset of at least one of U_α. Number r is called a Lebesque number of the cover.

SOLUTION:

Since $\{U_\alpha\}$ is a cover of X, each $x \in X$ belongs to at least one U_α. Each U_α is open, hence for each $x \in X$

$$x \in B(x, r_x) \subset U_\alpha. \tag{1}$$

The collection of sets

$$\left\{B\left(x, \frac{r_x}{2}\right) : x \in X\right\} \tag{2}$$

forms an open cover of X and since X is compact, a finite subcover of (2) exists

$$\left\{B\left(x_1, \frac{r_{x_1}}{2}\right), \dots, B\left(x_n, \frac{r_{x_n}}{2}\right)\right\}. \tag{3}$$

The Lebesque number is defined by

$$r = \min\left(\frac{r_{x_1}}{2}, \dots, \frac{r_{x_n}}{2}\right). \tag{4}$$

Indeed, let $x \in X$ then

$$x \in B\left(x_k, \frac{r_{x_k}}{2}\right)$$

for some $1 \le k \le n$. Let

$$y \in B(x, r) \tag{5}$$

then

$$d(y, x_k) \le d(y, x) + d(x, x_k) < r + \frac{r_{x_k}}{2} \le r_{x_k}. \tag{6}$$

Thus

$$B(x, r) \subset B(x_k, r_{x_k}) \subset U_\alpha \tag{7}$$

for some U_α.

● PROBLEM 16–39

Show that the function $f : R \to R$

$$f(x) = 2x \tag{1}$$

is uniformly continuous.

SOLUTION:

The function

$$f: X \to Y \tag{2}$$

from a metric space (X, d) into a metric space (Y, d') is said to be uniformly continuous if for every $\varepsilon > 0$, there is $\delta > 0$ such that for every $x \in X$

$$f(B(x, \delta)) \subset B(f(x), \varepsilon). \tag{3}$$

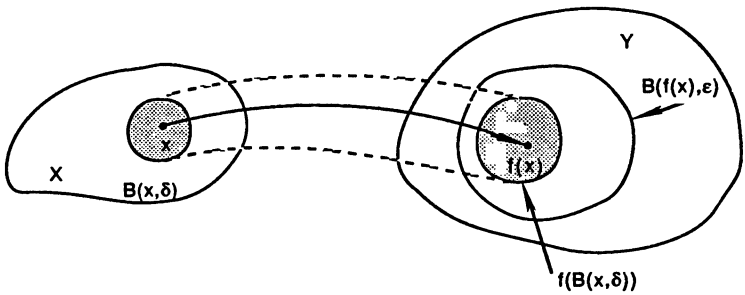


FIGURE 1

Let us choose any $\varepsilon > 0$ and set $\delta = \varepsilon/2$, then for any $x \in X$

$$f(B(x, \delta)) \subset B(f(x), \varepsilon). \tag{4}$$

Indeed let

$$y \in B(x, \delta)$$

then

$$|x - y| < \varepsilon/2$$

and

$$|f(x) - f(y)| = |2x - 2y| < 2 \cdot \varepsilon/2 = \varepsilon. \tag{5}$$

Thus, f is uniformly continuous.

● PROBLEM 16–40

Let $f: X \to Y$ represent a continuous function from a compact metric space (X, d) into a metric space (Y, d'). Show that f is uniformly continuous.

SOLUTION:

Let ε represent any positive number, $\varepsilon > 0$. We can define an open cover of Y as

$$\{B(y, {}^{\varepsilon}/_{2}) : y \in Y\}. \tag{1}$$

Since f is a continuous function, each $f^{-1}(B(y, {}^{\varepsilon}/_{2}))$ is an open set in X and

$$\{f^{-1}(B(y, {}^{\varepsilon}/_{2}))\} \tag{2}$$

is an open cover of X. But X is a compact metric space. Hence, by Problem 16–38, a Lebesque number r exists for cover (2).

Then for each $x \in X$,

$$f(B(x, q)) \subset B(f(x), \varepsilon). \tag{3}$$

Indeed, from the definition of the Lebesque number ($r = q$)

$$B(x, r) \subset f^{-1}(B(y, {}^{\varepsilon}/_{2})) \tag{4}$$

where $f(x) = y$

● PROBLEM 16–41

Let (R^2, d) represent R^2 space with the Pythagorean distance. Show that the subset $B(x, 1) \subset R^2$, where $x = (1, 1)$, is totally bounded.

SOLUTION:

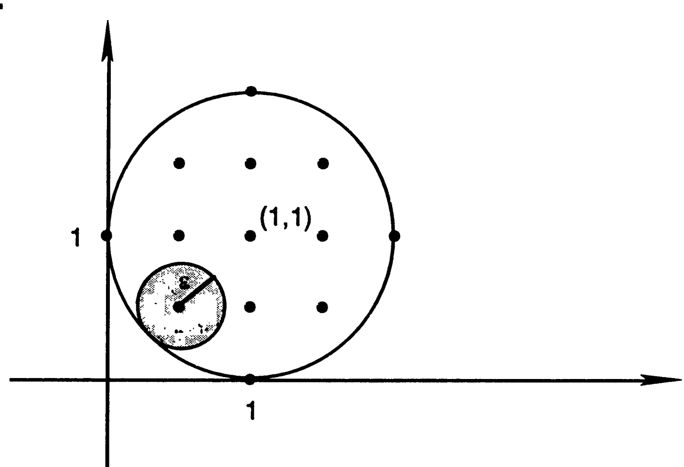


FIGURE 1

Let A represent a subset of a metric space (X, d) and let $\varepsilon > 0$. An ε–net for A is a finite set of points in X

$$N = \{x_1, x_2, \ldots, x_n\}$$

such that for every $x \in A$, $x_k \in N$ exists, such that

$$d(x_1, x_k) < \varepsilon.$$

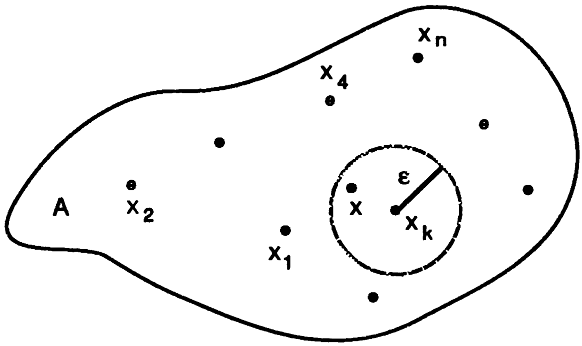


FIGURE 2

DEFINITION

A subset A of a metric space (X, d) is totally bounded if A has an ε-net for every $\varepsilon > 0$.

■

Let $\varepsilon > 0$. As an ε-net for a circle $B(x, 1)$ with the center at $x = (1, 1)$ and radius 1, we can choose all points inside the circle $B(x, 1)$ with coordinates

$$(1, 1), (1, 1 \pm \varepsilon), (1, 1 \pm 2\varepsilon), \ldots, (1 \pm \varepsilon, 1), \ (1 \pm 2\varepsilon, 1), \ \ldots,$$

$$\ldots, (1 \pm k\varepsilon, 1 \pm n\varepsilon)$$

where n and k are positive integers. Note that we are not looking for the smallest number of elements to form an ε-net.

● PROBLEM 16-42

Prove:

THEOREM

Totally bounded sets are bounded.

■

SOLUTION:

The diameter of A, $\delta(A)$, is defined by

$$\delta(A) = \sup\{d(x, x') : x, x' \in A\}. \quad (1)$$

Set A is bounded if

$$\delta(A) < \infty \tag{2}$$

Suppose A is a totally bounded set and let

$$N = \{x_1, x_2, \ldots, x_n\} \tag{3}$$

be ε-net for A.

Let $x, x' \in A$ be any elements of A. Then two elements of N exist, x_l and x_m, such that

$$d(x_1, x_l) < \varepsilon \quad d(x', x_m) < \varepsilon. \tag{4}$$

Also for any $x_l, x_m \in N$

$$d(x_l, x_m) < p < \infty \tag{5}$$

where p is some positive finite number.

We have

$$d(x, x') \le d(x, x_l) + d(x_l, x') \le$$

$$\le d(x, x_l) + d(x', x_m) + d(x_m, x_l) <$$

$$< \varepsilon + \varepsilon + p < \infty. \tag{6}$$

Hence

$$\delta(A) = \sup\{d(x, x') : x, x' \in A\} < \infty \tag{7}$$

and A is bounded.

● PROBLEM 16-43

Give an example of a bounded set which is not totally bounded.

SOLUTION:

We shall first define Hilbert space H. By R^∞ we denote the class of all infinite sequences

$$(a_1, a_2, \ldots) \tag{1}$$

such that

$$\sum_{n=1}^{\infty} a_n^2 < \infty. \tag{2}$$

Let $a, b \in R^\infty$ and

$$a = (a_1, a_2, \ldots) \quad b = (b_1, b_2, \ldots) \tag{3}$$

then the l_2–metric on R^∞ is defined by

$$d(a,b) = \sqrt{\sum_{n=1}^{\infty} |a_n - b_n|^2}. \tag{4}$$

The metric space (R^∞, d) with the l_2–metric is called Hilbert space, or l_2–space, and denoted by H. Let A represent a subset of H consisting of elements

$$e_1 = (1, 0, 0, \ldots)$$

$$e_2 = (0, 1, 0, \ldots)$$

$$e_3 = (0, 0, 1, \ldots)$$

Then

$$d(e_j, e_k) = \sqrt{1+1} = \sqrt{2} \tag{5}$$

and A is bounded because

$$\delta(A) = \sup\{d(e_j, e_k) : e_j, e_k \in A\} = \sqrt{2} < \infty. \tag{6}$$

But A is not totally bounded. Let $\varepsilon = {}^1/_2$ then, ε–net of A consists of all elements of A. The infinite set A cannot be separated into a finite number of subsets, each with diameter less than $\varepsilon = {}^1/_2$.

● PROBLEM 16–44

Show that sequentially compact subsets of a metric space are totally bounded.

SOLUTION:

Let A represent a subset of a metric space (X, d). We shall prove that if A is not totally bounded, then A is not sequentially compact. Suppose A is not totally bounded, then $\varepsilon > 0$ exists, such that no finite ε-net of A exists. Let $a_1 \in A$. See Figure 1.

Then $a_2 \in A$ exists, such that

$$d(a_1, a_2) \geq \varepsilon.$$

Otherwise $\{a_1\}$ would be ε-net. A point $a_3 \in A$ exists, such that

$$d(a_1, a_3) \geq \varepsilon$$

$$d(a_1, a_2) \geq \varepsilon$$

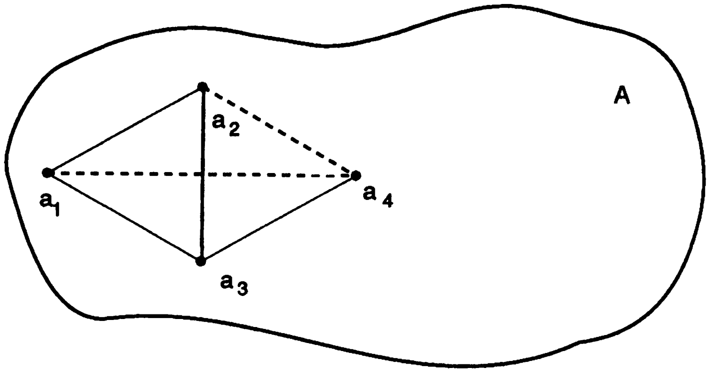


FIGURE 1

Otherwise $\{a_1, a_2\}$ would be ε-net. Following this procedure, we obtain a sequence of points

$$(a_1, a_2, a_3, \ldots),$$

such that

$$d(a_l, a_k) \geq \varepsilon \quad \text{for } l \neq k.$$

This sequence $(a_1, a_2, \ldots)$ does not have any convergent subsequence. Thus, A is not sequentially compact.

● PROBLEM 16–45

Show that if A is a countably compact subset of a metric space (X, d), then A is also sequentially compact.

SOLUTION:

Let

$$(a_1, a_2, \ldots) \tag{1}$$

represent a sequence in A. We shall show that a subsequence of (1) exists which converges to a point in A. If the set

$$\{a_1, a_2, \ldots\} \tag{2}$$

is finite then one of the elements, say a_l, of (2) appears infinitely many times in (1). Hence the subsequence

$$(a_l, a_l, a_l, \ldots) \tag{3}$$

of (1) converges to $a_l \in A$.

Suppose (2) is infinite. By hypothesis, A is countably compact. Hence, the infinite subset of A

$$\{a_1, a_2, a_3, \dots\}$$

has an accumulation point a in A. Since (X, d) is a metric space, we can choose a subsequence

$$(a_{n_1}, a_{n_2}, a_{n_3}, \dots) \tag{4}$$

of (1), which converges to $a \in A$. Thus, A is sequentially compact.

● PROBLEM 16–46

Prove this theorem:

THEOREM

If (X, d) is a metric space, then the following statements are equivalent:

1. X is compact.
2. X is countably compact
3. X is sequentially compact.

■

SOLUTION:

In Problem 16–28, we proved that any compact space is countably compact. Hence

$$1 \Rightarrow 2$$

for any metric space.

In Problem 16–45, we proved that any countably compact metric space is sequentially compact. Hence

$$2 \Rightarrow 3.$$

We shall prove that if a subset A of a metric space (X, d) is sequentially compact, then it is compact, i.e., $3 \Rightarrow 1$.

Let A be sequentially compact and let $\{U_\alpha\}$ be an open cover of A. Set $\{U_\alpha\}$ has a Lebesque number $r > 0$, since every open cover of a sequentially compact subset of a metric space has a Lebesque number. Since A is totally bounded, there is a decomposition of A into a finite number of subsets

$$B_1, B_2, \ldots, B_k$$

such that $d(B_1), d(B_2), \ldots, d(B_k) < r$. But r is a Lebesque number of A, hence open sets exist such that

$$B_1 \subset U_{\alpha_1}, \ldots, B_k \subset U_{\alpha_k}.$$

Thus

$$A \subset B_1 \cup \ldots \cup B_k \subset U_{\alpha_1} \cup \ldots \cup U_{\alpha_k}$$

and $\{U_{\alpha_1}, \ldots, U_{\alpha_k}\}$ is a finite subcover of $\{U_\alpha\}$. A is compact and

$$1 \Leftrightarrow 2 \Leftrightarrow 3.$$

● PROBLEM 16-47

Show that any countably compact metric space (X, d) is separable.

SOLUTION:

Let s represent any positive number. For any s, there is a maximal subset A_s of X, such that for any $a, b \in A_s$

$$d(a, b) \geq s. \tag{1}$$

Suppose A_s is infinite for some $s > 0$, then A_s has an accumulation point y. Then $B(y, {}^s/_2)$ contains infinitely many points of A_s. But any of these points are closer than s. Thus, a contradiction and A_s is finite for each s. But then for any $x \in X$

$$A_s \cap B(x, s) \neq \phi. \tag{2}$$

Otherwise we would obtain a contradiction to the maximality of A_s. For each positive integer n we find $A_{\frac{1}{n}}$. Then

$$\bigcup_{n \in N} A_{\frac{1}{n}}$$

is a countable dense subset of X. Hence X is separable.

● PROBLEM 16-48

Find an example of a compactification of the open interval (0, 1) with the absolute value topology.

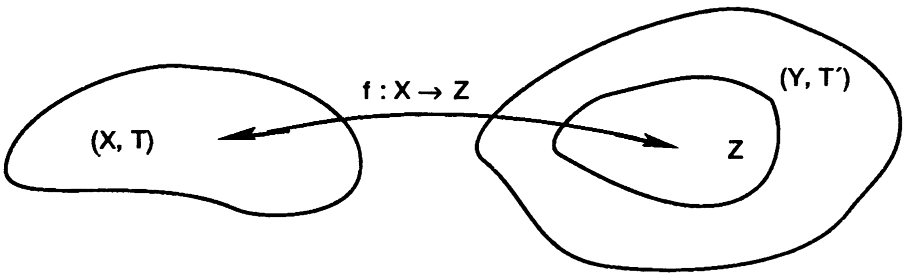
SOLUTION:

DEFINITION OF COMPACTIFICATION

Let (X, T) represent any topological space. A compactification of X is a compact space (Y, T'), such that X is homeomorphic to a dense subspace of Y.

■

Note that by compactification of X, we understand a space Y and a homeomorphism from X onto a dense subspace of Y.



f is homeomorphism
Z is dense in Y

FIGURE 1

Thus, for a given X and Y, it is possible to find more than one compactification (possibly infinitely many).

Two possible compactifications of the open interval $(0, 1)$ are the circle and the closed interval $[0, 1]$.

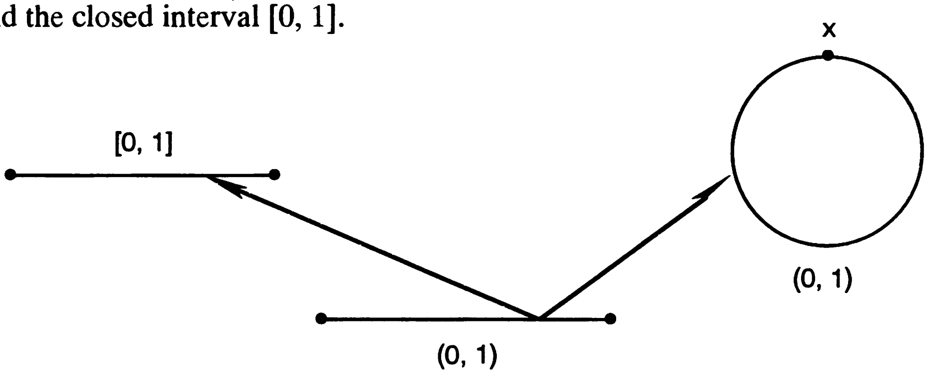


FIGURE 2

Obviously $(0, 1)$ is homeomorphic to a dense subset of $[0, 1]$. Also $(0, 1)$ is homeomorphic to a circle with one point removed.

● PROBLEM 16–49

Let P denote the (x, y) plane in the R^3 space with the Euclidean topology. Let S be the sphere of radius 1 and center at (0, 0, 1).

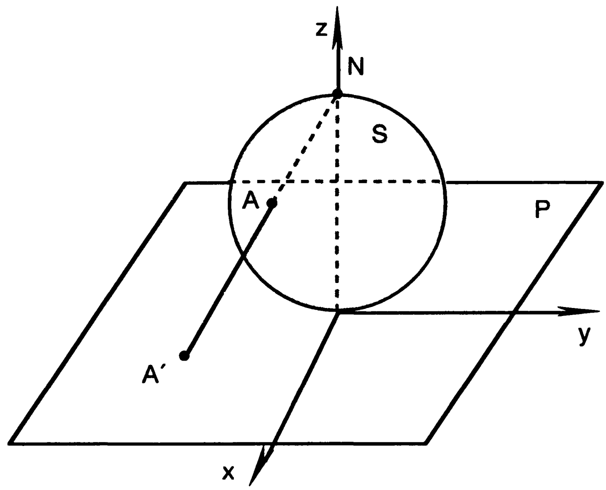


FIGURE 1

Show that S is a compactification of P.

SOLUTION:

Consider a line passing through the point $N = (0, 0, 2)$ on the sphere S and any point A on the sphere as shown in the figure.

Take any point A' on the plane, $A' \in P$. Then a line from A' to N determines the image of A' on the sphere.

$$f : P \to S$$

is defined by

$$f(A') = A.$$

f is one-to-one and onto and both f and f^{-1} are continuous. Hence, f is a homeomorphism from the plane P onto the subset $S - \{N\}$ of S. Obviously P is not compact, while S is compact.

The set $S - \{N\}$ is dense in S. Thus, S is a compactification of P.

● PROBLEM 16–50

Define the Alexandroff (sometimes called one-point) compactification of a T_2–space.

SOLUTION:

Let (X, T) represent any T_2–space. If X is compact then we define the Alexandroff compactification of X to be X itself.

Now suppose X is not compact. Let A be any point which does not belong to X. We define

$$Y = X \cup \{A\}$$

and the topology on Y as follows: U is open in Y if U is open in X, i.e., $U \in T$; or if $A \in U$, then $Y - U$ is a compact subset of X.

The Alexandroff, or one-point, compactification of X is defined as the set Y with the topology defined above.

Point A is called an ideal point. It can be shown that

1. Y is a topological space
2. Y is compact
3. X is dense in Y.

It can be also proved that:

THEOREM

The Alexandroff compactification Y of any topological T_2–space, (X, T) is a topological space and it is a compactification of X in the sense of the definition of Problem 16–48.

● PROBLEM 16–51

Prove the following:

THEOREM

The Alexandroff compactification of a space (X, T) is T_2 if and only if X is T_2 and locally compact.

■

SOLUTION:

Suppose (X, T) is compact. Then the Alexandroff compactification of X is X itself.

Then if X is T_2, X is T_2 and also locally compact. Also if X is T_2 and locally compact, then X is T_2.

Suppose X is not compact and Y is its Alexandroff compactification. If Y

is T_2, then X is T_2, since X is a subspace of Y.

Since Y is T_2 and compact, Y is also locally compact. X is locally compact as an open subspace of Y.

Now suppose X is T_2 and locally compact. Let x and y represent elements of Y, $x \neq y$. If $x, y \in X$, then open sets U and V exist such that

$$x \in U \quad \text{and} \quad y \in V$$

$$U \cap V = \phi$$

because X is T_2.

Suppose $x = A$ (A is an ideal point). Then $y \in X$ and a compact subset B of X exists, such that

$$y \in \mathit{Int}\, B \subset B.$$

Hence $Y - B$ is a neighborhood of $x = A$ and $\mathit{Int}\, B$ is a neighborhood of y and

$$(Y - B) \cap \mathit{Int}\, B = \phi.$$

Thus Y is T_2.

CHAPTER 17

CONNECTEDNESS

Show that any set X, consisting of two or more points, with discrete topology is disconnected.

SOLUTION:

DEFINITION OF MUTUAL SEPARATION

In the space (X, T), two nonempty, disjoint sets $U \subset X$, $V \subset X$ are said to be mutually separated if neither U, nor V contains a boundary point of the other.

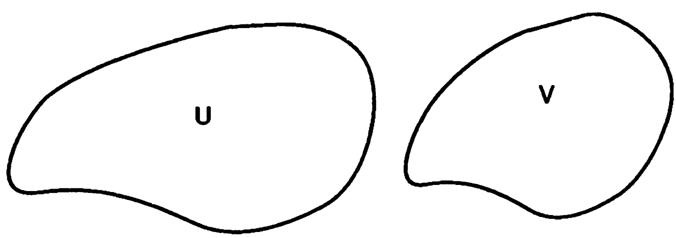
■

DEFINITION OF CONNECTED SPACES

A space (X, T) is said to be disconnected if the set X can be expressed as the union of multiple, mutually separated, nonempty subsets of X. The set X is connected if it is not disconnected, (see Figure 1).

■

X = U ∪ V



U ∩ V = ϕ

FIGURE 1

Space (X, T) with the discrete topology consisting of more than one point is disconnected. Indeed take any

$$x \in X \tag{1}$$

and consider the sets

$$\{x\} \text{ and } X - \{x\}. \tag{2}$$

Both sets (2) are open, nonempty, and disjoint. Their union is X

$$\{x\} \cup [X - \{x\}] = X. \tag{3}$$

Any set with the trivial topology is connected since the only nonempty open subset of X is X itself.

Show that connectedness is a topological property.

SOLUTION:

We shall prove that if $f: (X, T) \to (Y, T')$ is a continuous function onto Y and X is connected, then Y is connected.

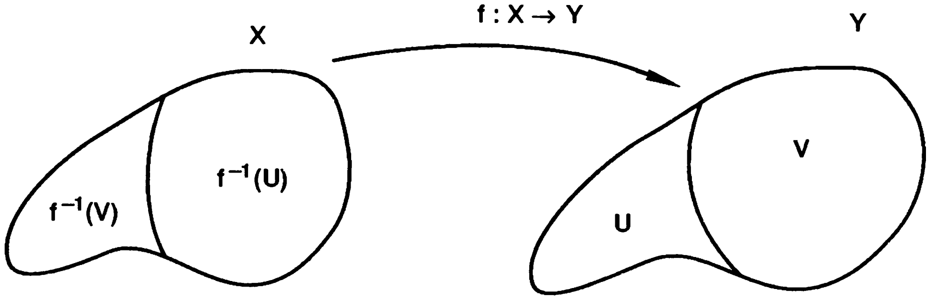


FIGURE 1

Suppose Y is not connected. Then there are open sets U and V such that

$$Y = U \cup V, \quad U \cap V = \phi. \tag{1}$$

Since f is continuous, then both $f^{-1}(U)$, and $f^{-1}(V)$ are open subsets of X. Since f is a bijection then

$$f^{-1}(U) \cap f^{-1}(V) = \phi \tag{2}$$

and

$$f^{-1}(U) \cup f^{-1}(V) = X. \tag{3}$$

Hence X is disconnected which is a contradiction. Therefore, Y is connected.

For example, if X is a connected space and R is an equivalence relation on R, then the identification space $X/_R$ is connected.

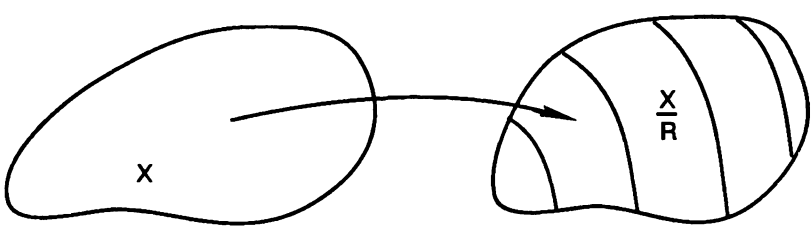


FIGURE 2

The identification mapping f: X $\to X/_R$ is continuous.

● PROBLEM 17-3

Show that if (X, T) is any topological space, then the following statements are equivalent:

1. X is connected.

2. X cannot be expressed as the union of two nonempty, disjoint, closed subsets.

SOLUTION:

1. $\Rightarrow$ 2.

Suppose X is connected and two sets F and G exist such that

$$X = F \cup G \text{ and } F \cap G = \phi \tag{1}$$

and F, G are nonempty and closed in X. Then

$$G = X - F$$

$$F = X - G \tag{2}$$

as complements of closed sets are open. Hence,

$$X = F \cup G$$

is the union of disjoint, nonempty, open subsets of X. Thus, X is not connected.

2. $\Rightarrow$ 1.

Suppose X is disconnected. Then open sets U, V exist such that

$$X = U \cup V, \quad U \cap V = \phi \tag{3}$$

where U, V are nonempty. Then

$$U = X - V \quad \text{and} \quad V = X - U.$$

Hence, U and V are closed, a contradiction with 2.

● PROBLEM 17-4

Let (X, T) be any topological space. Show that the following statements are equivalent:

1. X is connected.

2. The only subsets of X which are open and closed are X and ϕ.

3. No continuous

$$f : X \to Y$$

is onto, where $Y = \{0, 1\}$ with the discrete topology.

SOLUTION:

1. $\Rightarrow$ 2.

Suppose $A \subset X$ is both open and closed, then

$$A \text{ and } X - A$$

are both open and

$$A \cup (X - A) = X, \quad A \cap (X - A) = \phi.$$

Hence, X is disconnected, a contradiction.

2. $\Rightarrow$ 3.

Suppose

$$f : X \to \{0, 1\}$$

is a continuous and onto. Then

$$f^{-1}(0) \neq \phi, \; f^{-1}(0) \neq X.$$

Since the subset $\{0\}$ of $\{0, 1\}$ is open and closed,

$$f^{-1}(0)$$

is open and closed in X. Contradiction.

3. $\Rightarrow$ 1.

Let U and V be disjoint, nonempty, open sets, such that

$$X = U \cup V.$$

The U and V are also closed. The characteristic function

$$\chi_A : X \to \{0, 1\}$$

is continuous and onto.

● PROBLEM 17-5

Suppose (X, T) is a space such that

$$X = U \cup V$$

$$U \cap V = \phi \tag{1}$$

where U and V are open, nonempty subsets of X. Show that if A is any connected subset of X, then either

$$A \subset U \quad \text{or} \quad A \subset V. \tag{2}$$

SOLUTION:

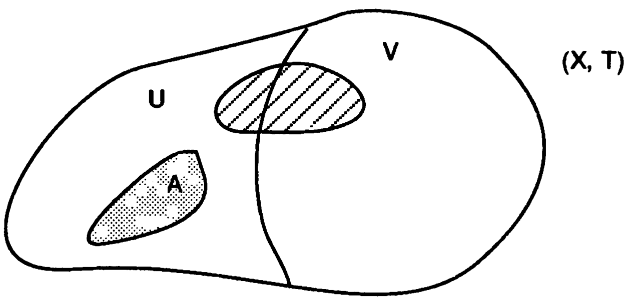


FIGURE 1

Suppose

$$A \cap U \neq \phi \quad \text{and} \quad A \cap V \neq \phi. \tag{3}$$

Then sets $A \cap U$ and $A \cap V$ are nonempty, disjoint subsets of A which are open in A. Then

$$A = (A \cap U) \cup (A \cap V). \tag{4}$$

Hence, A is not connected.

Therefore, either $A \cap U = \phi$ or $A \cap V = \phi$. Hence, we have

$$A \subset V \quad \text{or} \quad A \subset U.$$

Note that a subset A of X is said to be connected if the subspace A is connected.

● PROBLEM 17-6

Let A and B be connected sets which are not mutually separated. Show that

$$A \cup B$$

is connected.

SOLUTION:

For working convenience write the definition of mutually separated sets, as given in Problem 17–1, in terms of set theory.

DEFINITION OF MUTUALLY SEPARATED SETS

In the topological space (X, T), $A \subset X$ and $B \subset X$ are said to be mutually separated if

$$A \cap B = \phi$$

$$\overline{A} \cap B = \phi$$

$$A \cap \overline{B} = \phi$$

■

Suppose $A \cup B$ is disconnected. Then

$$A \cup B = U \cup V.$$

Since A is a connected subset of $A \cup B$, we have either

$$A \subset U \quad \text{or} \quad A \subset V$$

by Problem 17–5.

Also, either

$$B \subset U \quad \text{or} \quad B \subset V.$$

If $A \subset U$ and $B \subset V$ (or $A \subset V$ and $B \subset U$), then

$$A = (A \cup B) \cap U$$

$$B = (A \cup B) \cap V$$

are separated sets, a contradiction with the hypothesis. Thus, either

$$A \cup B \subset U \quad \text{or} \quad A \cup B \subset V.$$

Therefore, $U \cup V$ is not a disconnection of $A \cup B$ and $A \cup B$ is connected.

● PROBLEM 17-7

Prove the following:

THEOREM

The union of any family of connected subsets of any space (X, T) having at least one point in common is also connected.

■

SOLUTION:

Let $\{A_\alpha\}$ be a family of connected subsets of X such that for each α

$$x_0 \in A_\alpha$$

$$A = \bigcup_\alpha A_\alpha$$

$$x_0 \in \bigcap_\alpha A_\alpha$$

Let (Y, T') be the space $Y = \{0, 1\}$ with discrete topology. Let

$$f : A \to Y$$

be continuous. Since each A_α is connected and each $f \mid A_\alpha$ is continuous, no $f \mid A_\alpha$ is onto. Because for each α

$$x_0 \in A_\alpha$$

we obtain

$$f(x) = f(x_0)$$

for each $x \in A_\alpha$ for all α. Hence, f is not onto and $\bigcup_\alpha A_\alpha$ is connected.

Note that the intersection of two connected sets need not be connected.

● PROBLEM 17-8

Let A be a connected subset of (X, T) and let

$$A \subset B \subset \overline{A} \qquad (1)$$

Show that B is connected.

SOLUTION:

Suppose B is disconnected, then

$$B = U \cup V.$$

Since A is a connected subset of B, we have either

$$A \cap U = \phi \quad \text{or} \quad A \cap V = \phi.$$

Suppose $A \cap U = \phi$. Then $X - U$ is closed and

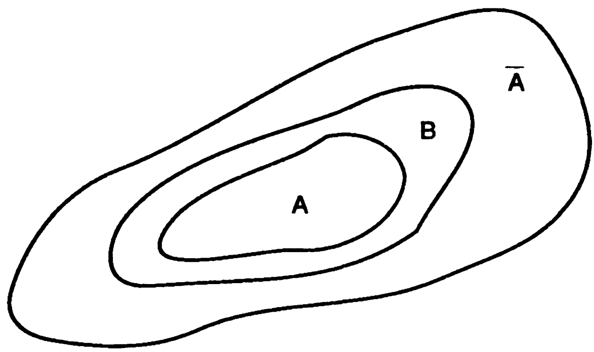


FIGURE 1

$$A \subset B \subset \bar{A} \subset X - U.$$

Thus,

$$B \cap U = \phi.$$

This is in contradiction with $B = U \cup V$. Hence, B is connected.

● PROBLEM 17-9

Show that the only connected subsets of R with the absolute value topology, having more than one point, are R and the intervals (open, closed, half-open).

SOLUTION:

We shall show that

$$(Y \text{ connected subset of } R) \Leftrightarrow (Y \text{ is an interval}).$$

$\Rightarrow$ Suppose Y is connected and is not an interval. Then there are

$$a, b \in Y \quad \text{and} \quad c \notin Y \tag{1}$$

such that $a < c < b$. Then

$$Y \cap \{x : x > c\}$$

$$Y \cap \{x : x < c\} \tag{2}$$

is a decomposition of Y, since both sets (2) are open, disjoint, and nonempty.

$\Leftarrow$ Suppose Y is an interval and is not connected. Then disjoint, nonempty, open sets A, B exist such that

$$Y = A \cup B \tag{3}$$

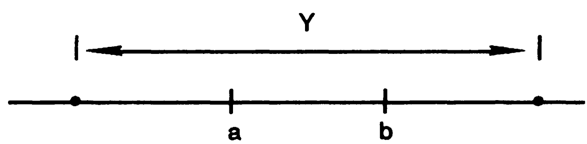


FIGURE 1

we can find $a \in A$, $b \in B$, such that

$$a < b. \tag{4}$$

Let us define

$$\omega = \sup \{x : [a, x) \subset A\} \tag{5}$$

and since Y is an interval,

$$\omega \in Y \quad \omega \leq b. \tag{6}$$

We have

$$\omega \in \overline{A_Y} \tag{7}$$

and since

$$A = Y - B$$

is closed in Y, we have

$$\omega \in A. \tag{8}$$

But, A is also open in Y, hence, $\varepsilon > 0$ exists such that

$$(\omega - \varepsilon, \omega + \varepsilon) \subset A \tag{9}$$

which contradicts the definition of ω.

Prove the following variation of the fixed-point theorem:

THEOREM 1

If $f: [0, 1] \to [0, 1]$ is continuous, then $x_0 \in [0, 1]$ exists such that

$$f(x_0) = x_0.$$

■

SOLUTION:

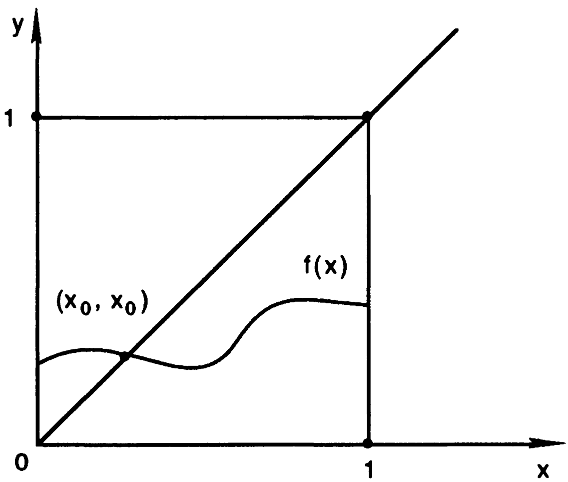


FIGURE 1

In Problem 17–2, we proved that a continuous image of a connected set is connected. The only connected subsets (consisting of more than one point) of R are intervals. Hence, the conclusion in the form of a theorem.

THEOREM 2

If $f: X \to R$ is a continuous function defined on a connected set X, then f assumes all values between any two of its values.

■

Theorem 1 is a conclusion of Theorem 2. Also note that the graph of the continuous function

$$f: [0, 1] \to [0, 1]$$

is located in the square

$$[0, 1] \times [0, 1].$$

Hence, it must intersect the diagonal joining (0, 0) with (1, 1). Therefore, point $x_0 \in [0, 1]$ exists such that

$$f(x_0) = x_0.$$

● PROBLEM 17-11

Consider Theorem 1 of Problem 17–10. Prove this theorem using the sets

$$U = \{(x, y) : x < y\}$$

$$V = \{(x, y) : x > y\}. \tag{1}$$

Observe that the graph of f is connected.

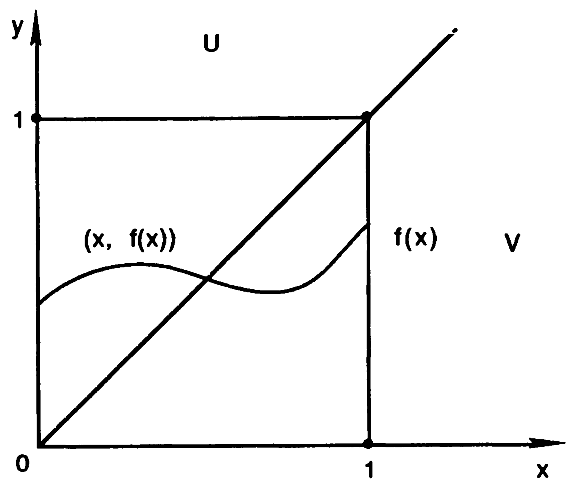


FIGURE 1

SOLUTION:

If $f(0) = 0$ or $f(1) = 1$, the theorem is proved. Otherwise

$$f(0) > 0 \quad \text{and} \quad f(1) < 1 \tag{2}$$

then

$$(0, f(0)) \in U \quad (1, f(1)) \in V. \tag{3}$$

Suppose the graph of f

$$F = \{(x, y) : x \in [0, 1], y = f(x)\} \tag{4}$$

does not contain any points of the diagonal

$$D = \{(x, y) : x = y\}. \tag{5}$$

Then $U \cup V$ is a disconnection of F. But F is connected as a continuous image of a connected set. Contradiction. Thus, F contains a point (x_0, x_0) of the diagonal, hence

$$f(x_0) = x_0. \tag{6}$$

● PROBLEM 17-12

Let (X, T) be a space such that for any two elements

$$x, y \in X \tag{1}$$

a connected set $A \subset X$ exists such that

$$x, y \in A. \tag{2}$$

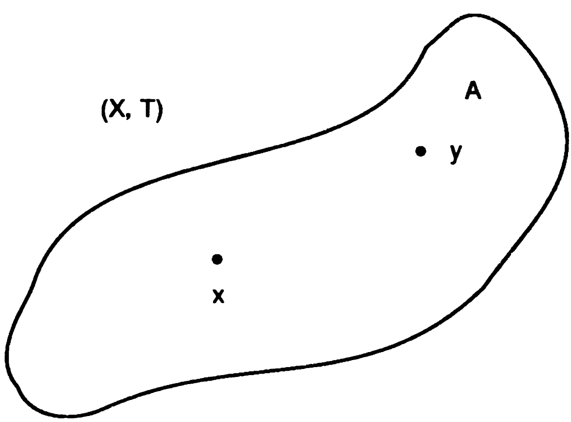


FIGURE 1

Show that (X, T) is connected.

SOLUTION:

Let x be a fixed element of X. Then, for any $y \in X$ there exists a connected subspace $A(x, y)$ of X such that

$$x, y \in A(x, y).$$

Consider the family of sets

$$\{A(x, y) : y \in X\} \tag{3}$$

which consists of connected subspaces of X and

$$X = \bigcup_{y \in X} A(x,y). \tag{4}$$

The intersection of (3) is nonempty

$$X \in \bigcap_{y \in X} A(x,y). \tag{5}$$

By Problem 17–7, we conclude that (X, T) is connected.

● PROBLEM 17–13

Apply Problem 17–12 to show that the Euclidean n-space R^n ($n > 1$) with one point S removed

$$R^n - \{S\}$$

is connected.

SOLUTION:

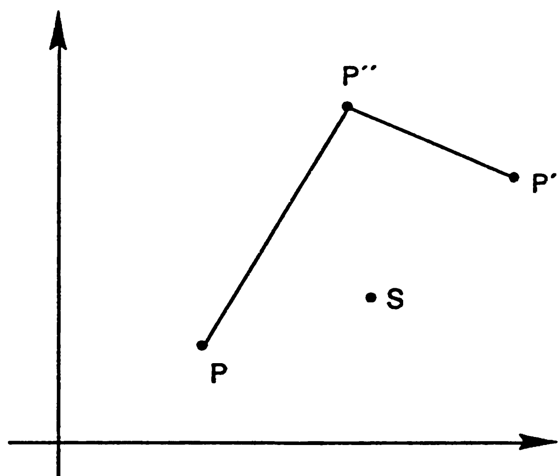


FIGURE 1

Suppose P and P' are any two points of $R^n - \{S\}$. Any closed line segment in the Euclidean space R^n is homeomorphic to the closed interval [0, 1]. Hence, it is a connected subspace of R^n. We choose point $P'' \in R^n - \{S\}$ such that

$$S \notin \overline{PP''}$$

and

$$S \notin \overline{P''P'}.$$

The sets $\overline{PP''}$ and $\overline{P''P'}$ are connected and their intersection is nonempty

$$\overline{PP''} \cap \overline{P''P'} = \{P''\} \neq \phi.$$

Hence,

$$\overline{PP''} \cup \overline{P''P'}$$

is connected.

Points P and P' belong to the connected subspace of $R^n - \{S\}$. Hence, $R^n - \{S\}$ is connected. Note that this is not true for $n = 1$.

● PROBLEM 17-14

Which of the subsets of R^2 are polygonally connected?

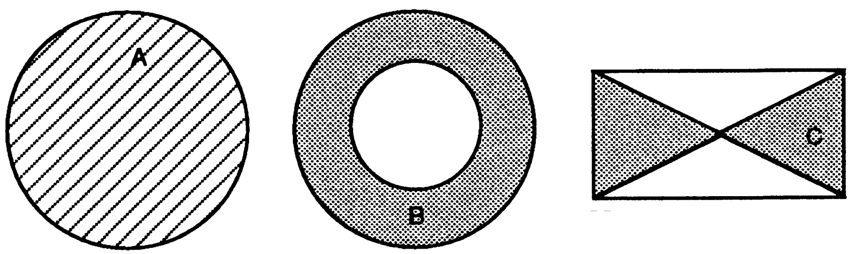


FIGURE 1

1. A
2. B
3. C
4. $A \cup B$

SOLUTION:

DEFINITION OF A PATH AND PATH-CONNECTED SPACES

A subspace Y of any space (X, T) is said to be a path in X if there is a continuous function from $[0, 1]$ (with the absolute value topology)

$$f: [0, 1] \rightarrow Y$$

onto Y.

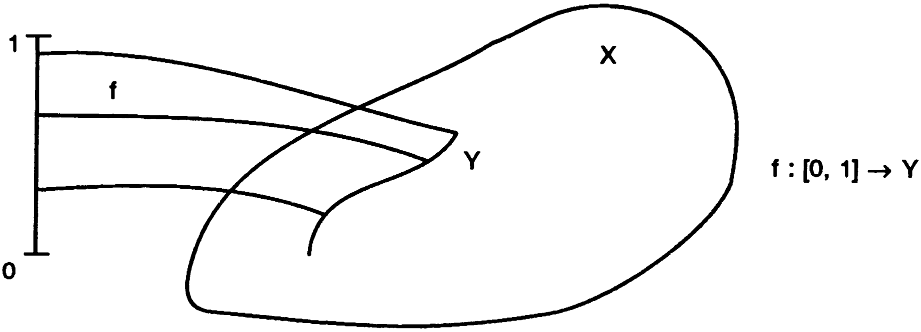


FIGURE 2

X is said to be path-connected if any two points x, y of X belong to some path in X.

■

Consider the Euclidean space R^n. A subset $A \subset R^n$ is said to be polygonally connected if for any $x, y \in A$ there are points

$$x = x_0, x_1, \ldots, x_{k-1}, x_k = y$$

such that

$$\bigcup_{i=0}^{k-1} \overline{x_i x_{i+1}} \subset A$$

where $\overline{x_i x_{i+1}}$ is a closed segment joining x_i and x_{i+1}.

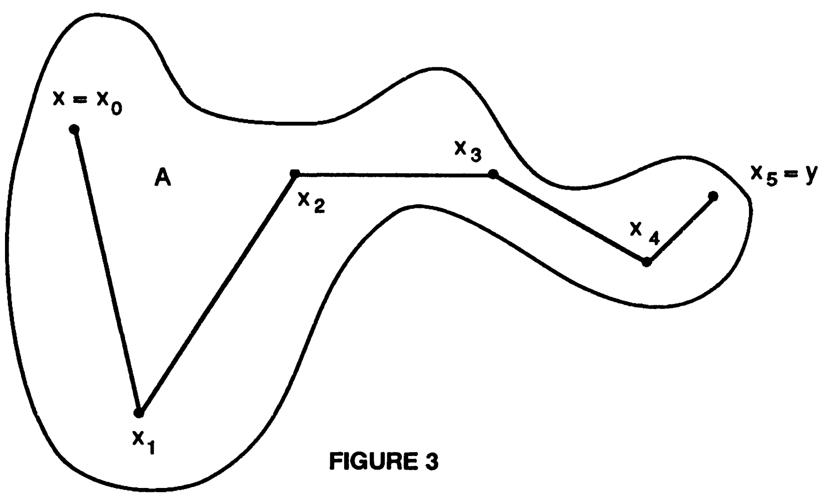


FIGURE 3

Hence, A, B, and C are polygonally connected, while $A \cup B$ is not.

1. Show that a subset of R^n, which is path-connected, does not have to be polygonally connected.

2. Give an example of a connected subspace of R^2 which is not path-connected.

SOLUTION:

1. The circle

$$\{(x, y) : x^2 + y^2 = 1\} \quad R^2$$

is path-connected, but is not polygonally connected.

2. Consider the set

$$A = \{(x, y) : y = \sin {}^1/_x, x > 0\} \cup \{0, 0\}$$

which is a subset of R^2.

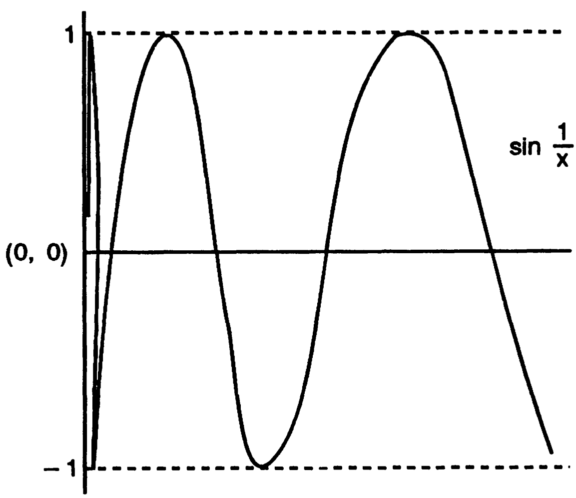


FIGURE 1

From Problem 17–8 we see that A is connected.

But A is not path-connected. There is no path in A which contains (0, 0) and any point $P \neq (0, 0)$ of A. Otherwise the function $f(x) = \sin {}^1/_x$, $x > 0$ would be continuous.

● PROBLEM 17-16

Prove that any polygonally connected subspace of R^n is path-connected.

SOLUTION:

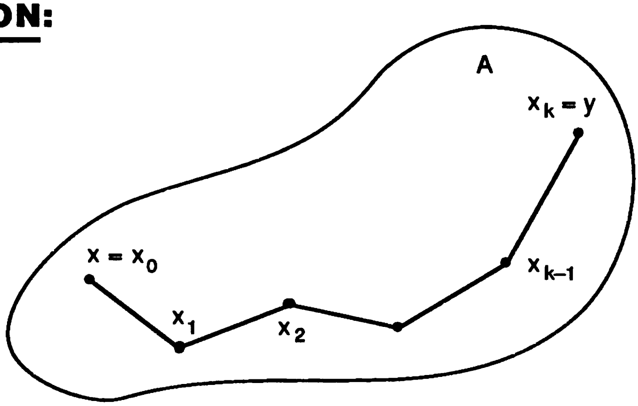


FIGURE 1

Let $A \subset R^n$ be polygonally connected and let x, y be any points in A.

Since A is polygonally connected, there are points $x = x_0, x_1, \ldots, x_k = y$ such that

$$\bigcup_{i=0}^{k-1} \overline{x_i x_{i+1}} \subset A$$

Set $\bigcup_{i=0}^{k-1} \overline{x_i x_{i+1}}$ is a path. A continuous function exists

$$f : [0,1] \rightarrow \bigcup_{i=0}^{k-1} \overline{x_i x_{i+1}}.$$

Hence, A is path-connected.

● PROBLEM 17-17

Let

$$f : [0, 1] \rightarrow X, f(0) = a, f(1) = b$$

be a path from $a \in X$ to $b \in X$ and

$$g : [0, 1] \rightarrow X, g(0) = b, g(1) = c$$

be a path from $b \in X$ to $c \in X$.

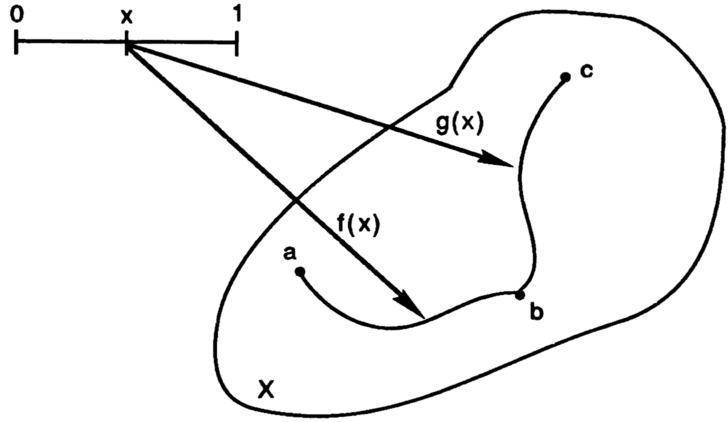


FIGURE 1

Show that the juxtaposition of the two paths f and g is a path from a to c.

SOLUTION:

The juxtaposition of the two paths f and g, denoted by $f \circ g$ is defined as follows:

$$f \circ g : [0, 1] \to X$$

$$(f \circ g)(t) = \begin{cases} f(2t) & \text{for} \quad 0 \le t \le \frac{1}{2} \\ g(2t-1) & \text{for} \quad \frac{1}{2} \le t \le 1 \end{cases} .$$

Note that $$(f \circ g)(0) = f(0) = a$$

$$(f \circ g)(^1/_2) = f(1) = b = g(0)$$

and $$(f \circ g)(1) = g(1) = c.$$

Thus, $f \circ g$ is a path from a to c.

● PROBLEM 17-18

1. Show that if (X, T) is a path-connected space, then X is connected.
2. Explain why any convex subset of R^n is connected.

SOLUTION:

1. In Problem 17–7 we proved that if (X, T) is a space and

$$X = \bigcup_{\alpha} A_\alpha$$

where $\{A_\alpha\}$ is a collection of connected subspaces of X and

$$\bigcap_{\alpha} A_\alpha \neq \phi$$

contained in some connected subspace of X, then X is connected.

Suppose X is path-connected and $x \in X$. Then for each $y \in X$, $P(x, y)$ denotes a path from x to y.

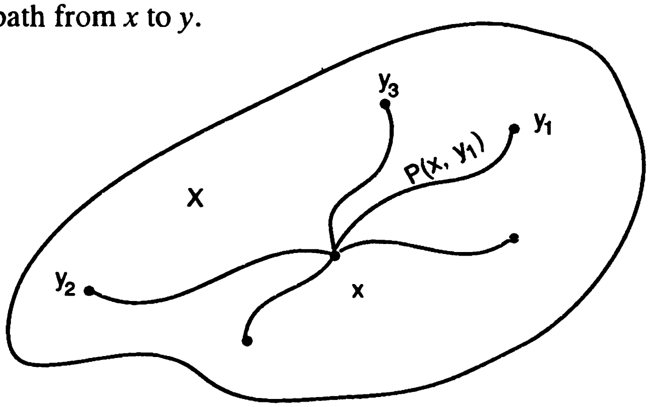


FIGURE 1

Then

$$X = \bigcup_{y \in X} P(x,y)$$

and

$$X \in \bigcap_{y \in X} P(x,y) \neq \phi.$$

Since each $P(x, y)$ is connected (as a continuous image of a connected set $[0, 1]$), X is connected.

2.

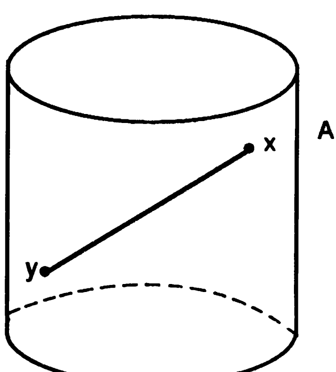


FIGURE 2

A subset A of R^n is convex if, for any points x and y of A, the closed segment $\overline{xy}$ is a subset of A.

Any convex subset of R^n is polygonally connected and, hence, by Problem 17–16, is path-connected and, hence, is connected.

● PROBLEM 17-19

Prove the following theorem concerning the Euclidean R^n spaces.

THEOREM

If U is an open connected subset of R^n, then U is polygonally connected. ■

SOLUTION:

Suppose $U \subset R^n$ is open and connected and $u \in U$.

Let X be all points of U which can be polygonally connected with u

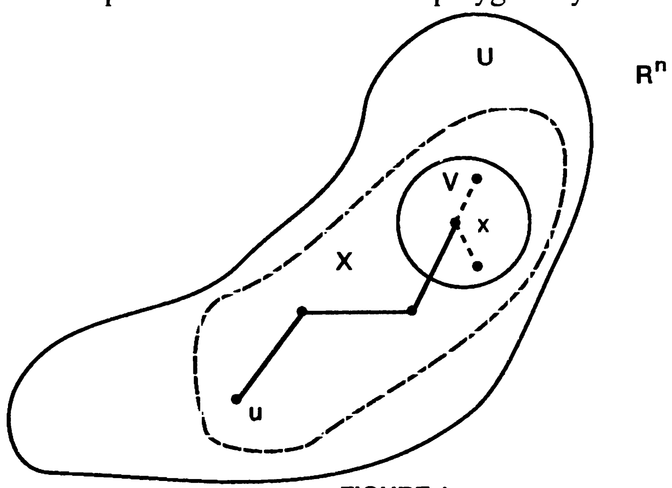


FIGURE 1

$$X = \{x \in U : x \text{ can be polygonally connected to } u \in U\}. \tag{1}$$

Let

$$Y = U - X. \tag{2}$$

Since U is open for any $x \in X$, there is an open neighborhood V of x such that u can be polygonally connected with any point of V; hence,

$$V \subset X. \tag{3}$$

Therefore, X is open.

The set $Y = U - X$ consists of elements which cannot be polygonally connected to u. By the same argument Y is open. Thus,

$$U = X \cup Y, X \cap Y = \phi \tag{4}$$

where X and Y are disjoint open subsets of U. Since U is connected, we have two options:

either $$X = \phi, \quad Y = U$$

or $$X = U, \quad Y = \phi. \tag{5}$$

Since $$u \in X, X \neq \phi$$

and $$X = U, \quad Y = \phi$$

and U is polygonally connected.

● PROBLEM 17-20

Show that for any $n > 1$, R^n is not homeomorphic to R.

SOLUTION:

Consider the space R (with the absolute value topology) with one point removed $R - \{a\}$.

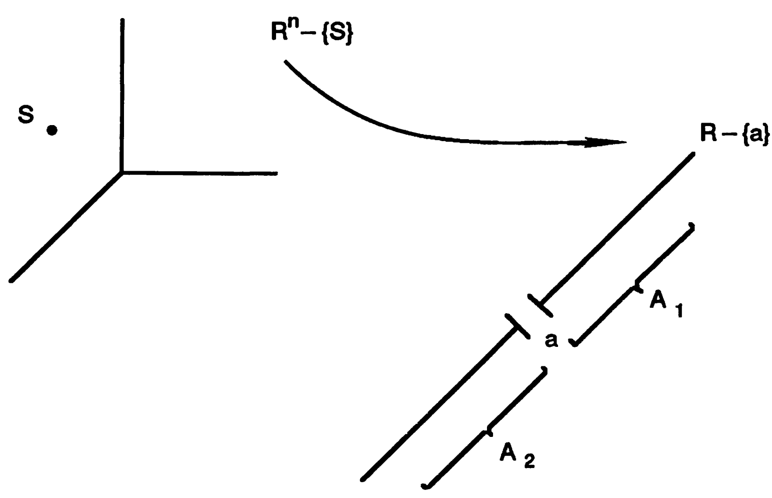


FIGURE 1

Then R is disconnected into two disjoint open sets

$$A_1 = \{x \in R : x > a\}$$

$$A_2 = \{x \in R : x < a\}$$

Hence, $R - \{a\}$ is disconnected. Suppose R^n $(n > 1)$ is homeomorphic to R, then for any point $S \in R^n$, $R^n - \{S\}$ is not connected.

In Problem 17–13, we showed that $R^n - \{S\}$ is connected. Therefore, R^n is not homeomorphic to R.

● PROBLEM 17–21

In Problem 17–18 we proved that

$$((X, T) \text{ is path-connected}) \Rightarrow ((X, T) \text{ is connected}).$$

Show that the converse of this theorem is not true.

SOLUTION:

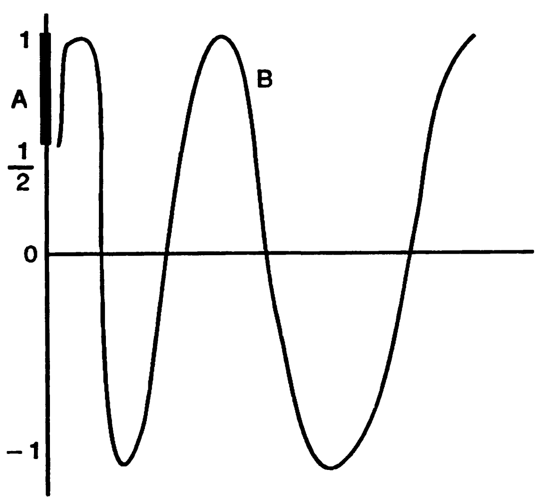


FIGURE 1

Let A and B be the subsets of R^2

$$A = \{(0, y) : {}^1/_2 \le y \le 1\}$$

$$B = \{(x, \sin {}^1/_x) : 0 < x \le 1\}.$$

Set A is a closed interval and B is a continuous image of a closed interval; hence, both are connected. Each point of A is an accumulation point of B, hence A and B are not separated. Thus,

$$A \cup B$$

is connected.

But $A \cup B$ is not path-connected. There is no path connecting any point of A with any point of B.

The converse implication is not true.

● PROBLEM 17-22

Show that the continuous image of a path-connected set is path-connected.

SOLUTION:

Let $A \subset X$ be path-connected and let $f : X \to Y$ be continuous. We shall show that $f(A)$ is path-connected. Let $a, b \in f(A)$, then $a', b' \in A$ (not necessarily uniquely defined) exist that

$$f(a') = a \quad \text{and} \quad f(b') = b.$$

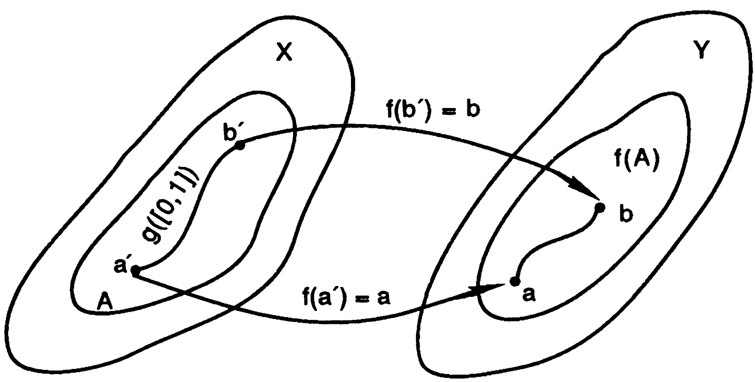


FIGURE 1

Since A is path-connected, there is a continuous function

$$g : [0, 1] \to A$$

such that $g(0) = a'$, $g(1) = b'$.

Since both f and g are continuous,

$$f \circ g : [0, 1] \to f(A)$$

is continuous and

$$f(g(0)) = f(a') = a$$

$$f(g(1)) = f(b') = b.$$

Thus, $f(A)$ is path-connected.

● PROBLEM 17-23

1. Show that an open disc in the Euclidean R^2 plane is path-connected (do not use results of Problem 17–19).

2. Let P be a class of path-connected subsets of X with a nonempty intersection

$$P = \{A_\alpha : \alpha \in A\}.$$

Show that

$$A = \bigcup_\alpha A_\alpha$$

is path-connected.

SOLUTION:

1. Let p and q be any points of an open disc in the R^2 plane.

$$p = (x_1, y_1)$$

$$q = (x_2, y_2)$$

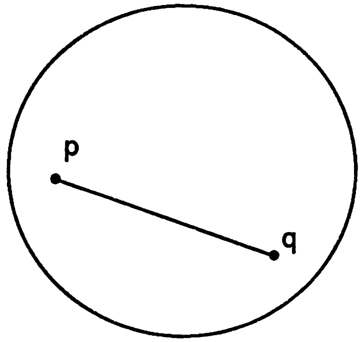


FIGURE 1

A path from p to q may be defined as

$$f : [0, 1] \to R^2$$

$$f(t) = (x_1 + t(x_2 - x_1), y_1 + t(y_2 - y_1))$$

for $t \in [0, 1]$. Then $f(t)$ is the line segment between p and q. For all points p and q in the open disc, $f(t)$ is a path contained completely in the disc. Therefore the open disc in R^2 is path-connected.

2. Let $x, y \in A$. Path-connected sets exist such that

$$x \in A_x, \quad y \in A_y$$

$$A_x,\ A_y \in P.$$

Nonempty intersection $\bigcap_\alpha A_\alpha$ contains at least one point, say q

$$q \in \bigcap_\alpha A_\alpha .$$

Then $x, q \in A_x$ and there is a path

$$f: [0, 1] \to A_x$$

$$f(0) = x, f(1) = q.$$

Similarly, there is a path

$$g: [0, 1] \to A_y$$

$$g(0) = q, \quad q(1) = y.$$

The juxtaposition of f and g is a path from x to y (see Problem 17–17). Hence, A is path-connected.

● PROBLEM 17–24

Let X be connected and let $P = \{A_\alpha\}$ be an open cover of X. Show that any pair of points of X can be joined by a simple chain consisting of elements of P.

SOLUTION:

DEFINITION OF A SIMPLE CHAIN

Subsets $A_1, A_2, \ldots, A_K$ of X are said to form a simple chain joining points x and y in X if

1. only A_1 contains x

2. only A_K contains y

3. $A_i \cap A_j = \phi$ iff $|i-j| > 1$

■

Let x be any point in X and let B be the set consisting of all points of X which can be joined to x by a simple chain consisting of elements of P.

Since $x \in B$, $B \neq \phi$. We shall prove that B is both open and closed, i.e., $B = X$ because X is connected.

B is open.

Let $b \in B$, then there is a simple chain $A_1, A_2, \ldots, A_l$ from x to b.

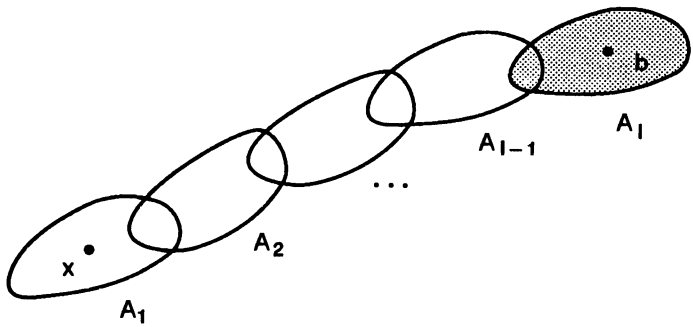


FIGURE 1

Note that all elements of A_l belong to B (chain $A_1, \ldots, A_l$ connects any element of A_l with x). Hence,

$$b \in A_l \subset B.$$

Since A_l is open, B is open.

B is closed.

Suppose

$$y \notin B.$$

Since P is a cover of X, there is an open $A_t \in P$ such that

$$y \in A_t$$

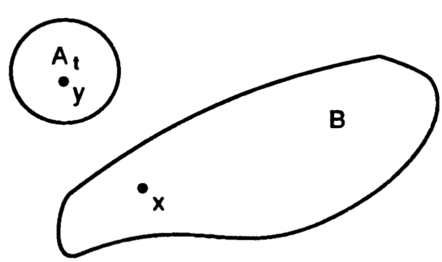


FIGURE 2

Suppose

$$A_t \cap B \neq \phi$$

then there is some chain from y to x and $y \in B$, contradiction.

Thus,

$$A_t \cap B = \phi$$

and

$$y \in A_t \subset X - B.$$

Hence, $X - B$ is open and B is closed. We conclude

$$X = B.$$

● PROBLEM 17-25

Let C_1 and C_2 be subsets of R^2

$$C_1 = \{(x, y) : x^2 + y^2 < 1\}$$

$$C_2 = \{(x, y) : (x-2)^2 + y^2 < 1\}$$

Are C_1 and C_2 mutually separated?

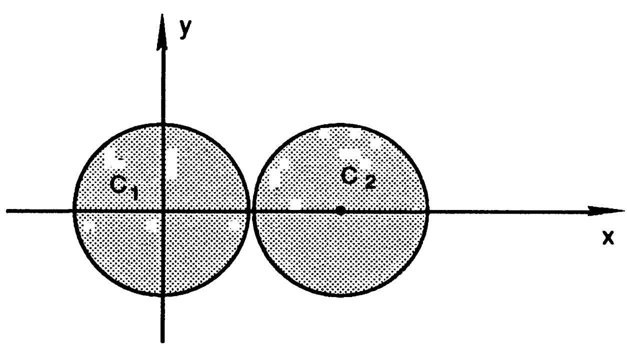


FIGURE 1

SOLUTION:

DEFINITION OF MUTUALLY SEPARATED SETS

Two sets A_1 and A_2 of a space (X, T) are said to be mutually separated if

$$\overline{A_1} \cap A_2 = A_1 \cap \overline{A_2} = \phi.$$

■

Where $\overline{A_1}$ indicates the closure of A_1. Hence,

$$\overline{C_1} = \{(x, y) : x^2 + y^2 \le 1\}$$

$$\overline{C_2} = \{(x, y) : (x-2)^2 + y^2 \le 1\}$$

and

$$\overline{C_1} \cap C_2 = C_1 \cap \overline{C_2} = \phi.$$

The sets C_1 and C_2 are mutually separated.

● PROBLEM 17–26

Not every subspace of a connected space is connected. The following theorem enables us to determine whether or not a subspace of a given space is connected.

THEOREM

A subspace A of (X, T) is connected iff A cannot be expressed as the union $C \cup D$, where C and D are nonempty, mutually separated subsets of X.

■

Prove it.

SOLUTION:

If A is not connected, then

$$A = C \cup D \quad \text{and} \quad C \cap D = \phi \tag{1}$$

where C and D are nonempty subsets of A, which are open and closed in A. Suppose

$$x \in C \cap \overline{D} \text{ then } x \in C \text{ and } x \in \overline{D}. \tag{2}$$

Since

$$C \subset A$$

$$x \in A \cap \overline{D} = \overline{D}_A = D. \tag{3}$$

Then

$$x \in C \cap D = \phi \tag{4}$$

a contradiction.

Hence,

$$C \cap \overline{D} = \phi. \tag{5}$$

Similarly, we show that

$$D \cap \overline{C} = \phi. \tag{6}$$

Now, suppose

$$A = C \cup D$$

$$C \cap \overline{D} = D \cap \overline{C} = \phi \quad \text{and} \quad C, D \neq \phi. \tag{7}$$

Then

$$\overline{C_A} = \overline{C} \cap A = (C \cup D) \cap \overline{C} =$$

$$= (C \cap \overline{C}) \cup (D \cap \overline{C}) = C \cup \phi = C. \tag{8}$$

Similarly,

$$\overline{D_A} = D \tag{9}$$

where $\overline{D_A}$ indicates the closure of D in A, i.e., $\overline{D_A} = \overline{D} \cap A$.

Hence, C and D are closed in A, and A is disconnected.

● PROBLEM 17–27

1. Show that if a space (X, T) contains a connected dense subspace, then X is connected.

2. Show that if (X, T) is connected then any compactification Y of X is connected.

SOLUTION:

1. Let $A \subset X$ be a connected dense subspace of X.

$$\overline{A} = X.$$

In Problem 17–8 we proved that if A is a connected subset of (X, T) and

$$A \subset B \subset \overline{A}$$

then B is connected.

Thus,

$$A \subset X \subset \overline{A}$$

and since A is connected, X is also connected.

2. Remember that a compactification of (X, T) is a compact space (Y, T') such that X is homeomorphic to a dense subspace of Y.

In other words, X is a dense connected subspace of Y; hence, Y is connected.

● PROBLEM 17–28

Let

$$X = \{(x, y) : 0 < x, y = \sin {}^1/_x \} \cup \{(0, 0)\} \subset R^2.$$

Apply Problem 17–8 to show that X is connected. (See following figure.)

SOLUTION:

Consider the set

$$A = X - \{(0, 0)\}.$$

We define a function

$$f : R \to R^2$$

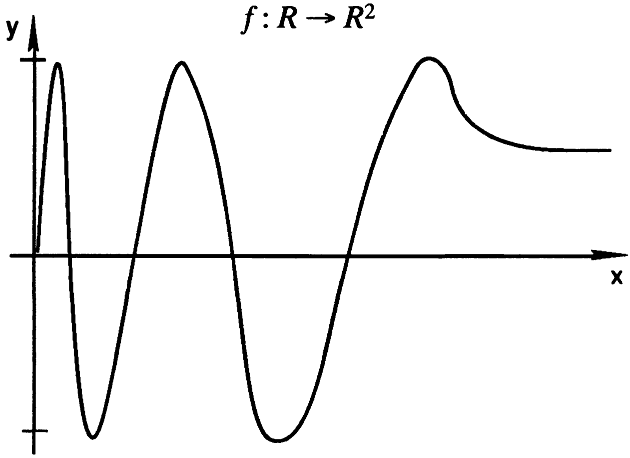


FIGURE 1

by

$$R \supset \{x : x > 0\} \ni x \to f(x) = (x, \sin {}^1/_x) \in R^2.$$

Note that f is continuous. Hence, since $\{x : x > 0\}$ is connected, $A = X - \{(0, 0)\}$ is connected as a continuous image of a connected set.

Since every neighborhood of $(0, 0)$ contains an infinite amount of points of A,

$$(0, 0) \in \overline{A}.$$

We have

$$A \subset X \subset \overline{A}.$$

Hence, since A is connected, by Problem 17–8, we conclude that X is connected.

● PROBLEM 17-29

Show that a component A of a space (X, T) is closed.

SOLUTION:

A disconnected space has connected subspaces.

DEFINITION OF COMPONENT

A component A of (X, T) is a maximal connected subset of X.

■

We see that A is connected and A is not a proper subset of any connected subset of X. By Problem 17–8 since

$$A \subset \overline{A} \subset \overline{A}$$

$\overline{A}$ is connected.

But

$$A \subset \overline{A}$$

and A is a maximal connected subspace of X; hence,

$$A = \overline{A}.$$

Therefore, A is closed.

● PROBLEM 17-30

1. Let R be the space of real numbers with the absolute value topology and let

$$A_+ = \{x \in R : x > a\}$$

$$A_- = \{x \in R : a > x\}$$

where $a \in R$. Find the components of $R - \{a\}$.

2. Let R^2 be the plane

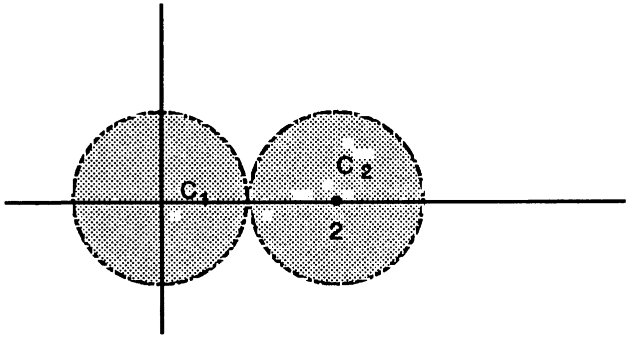


FIGURE 1

with the Pythagorean topology and

$$C_1 = \{(x, y) : x^2 + y^2 < 1\}$$

$$C_2 = \{(x, y) : (x-2)^2 + y^2 < 1\}$$

Find the components of the space $C_1 \cup C_2 \subset R^2$.

SOLUTION:

1. The components of

$$R - \{a\} = A_+ \cup A_-$$

are A_+ and A_-. Each is the maximal connected subspace of $R - \{a\}$.

2. Here the components of $C_1 \cup C_2$ are C_1 and C_2.

● PROBLEM 17-31

Show that each element of (X, T) is contained in a component of X.

SOLUTION:

Obviously each element x of X is contained in at least one connected subspace of X, namely $\{x\} \subset X$. We will show that there is a maximal connected subspace of X which contains x.

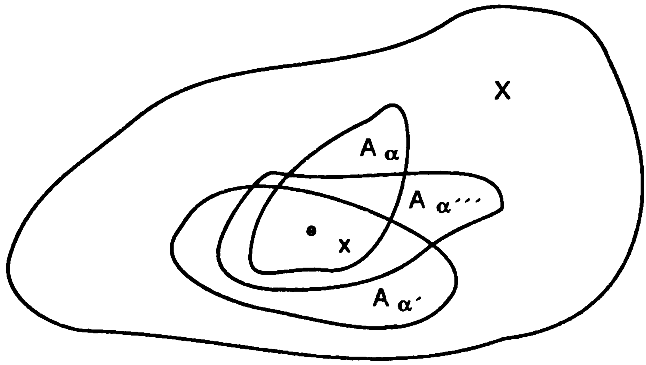


FIGURE 1

Let

$$\{A_\alpha : \alpha \in A\}$$

be the family of all connected subspaces of X which contains x,

$$x \in A_\alpha \quad X$$

and A_α is connected for each $\alpha \in A$. Then

$$\bigcup_\alpha A_\alpha$$

is a connected subspace of X which contains x. Indeed, since all A_α are connected and

$$x \in \bigcap_\alpha A_\alpha \neq \phi$$

then by Problem 17–7, $\bigcup_\alpha A_\alpha$ is connected.

$\bigcup_\alpha A_\alpha$ is a maximal connected subspace of X which contains x and is unique.

● PROBLEM 17–32

Let $\{A_\alpha\}$, $\alpha \in A$ be the set of components of a space (X, T). Show that the components of X form a partition of X, i.e., (see Figure 1 on following page)

$$X = \bigcup_\alpha A_\alpha$$

if $\alpha \neq \beta$, then $A_\alpha \cap A_\beta = \phi$.

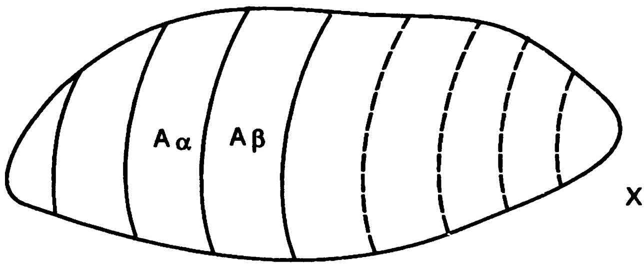


FIGURE 1

SOLUTION:

Any element x of X belongs to at least one connected subspace of X (for example, $x \in \{x\} \subset X$). Thus, a maximal connected subspace of X contains x. We have

$$X = \bigcup_{\alpha} A_{\alpha}.$$

We will show that if $\alpha \neq \beta$, then $A_\alpha \cap A_\beta = 0$. Suppose $\alpha \neq \beta$ and $A_\alpha \cap A_\beta \neq 0$, then $A_\alpha \cup A_\beta$ is a connected subspace of X which contains both A_α and A_β. A contradiction arises because A_α and A_β are maximal.

● PROBLEM 17–33

Prove that if a space has only a finite number of components, then each component is open.

SOLUTION:

Let

$$A_1, A_2, \ldots, A_K$$

be components of X.

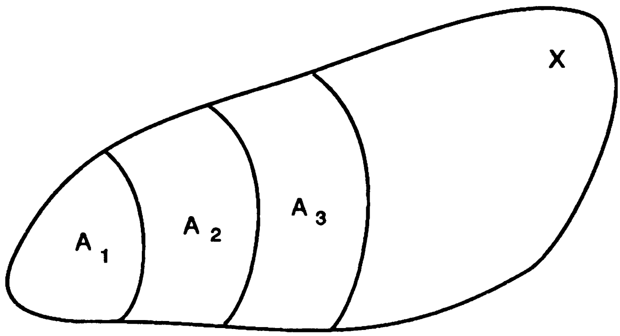


FIGURE 1

Then

$$X = A_1 \cup \ldots \cup A_K.$$

Since each pair of components are disjoint sets

$$A_i \cap A_j = \phi \quad \text{for } i \neq j$$

we have

$$A_1 = X - (A_2 \cup \ldots \cup A_K).$$

In Problem 17–29 we proved that a component of a space is closed. Therefore, since there are a finite number of components,

$$A_2 \cup \ldots \cup A_K$$

is a closed set. Hence, A_1 is an open set. Similarly, we show that all sets $A_1, A_2, \ldots, A_K$ are open.

● PROBLEM 17–34

Show that a finite product of connected spaces is connected.

SOLUTION:

Let X and Y be connected spaces and let

$$p = (x_1, y_1) \in X \times Y$$

$$q = (x_2, y_2) \in X \times Y. \tag{1}$$

The set

$$\{x_1\} \times Y$$

is homeomorphic to Y and since Y is connected, so is $\{x_1\} \times Y$. Similarly, we show that

$$X \times \{y_2\}$$

is connected since it is homeomorphic to X.

Therefore,

$$\{x_1\} \times Y \cup X \times \{y_2\}$$

is connected because

$$\{(x_1, y_2)\} = \{x_1\} \times Y \cap X \times \{y_2\} \neq \phi.$$

Also,

$$p, q \in \{x_1\} \times Y \cup X \times \{y_2\}.$$

Hence, p and q belong to the same component. But p and q were chosen arbitrarily; hence, $X \times Y$ has only one component — itself.

We conclude that a finite product of connected spaces is connected.

● PROBLEM 17-35

Show that a product of connected spaces is connected.

SOLUTION:

Let

$$\{X_\alpha : \alpha \in A\}$$

be a collection of connected spaces and let

$$X = \underset{\alpha \in A}{\times} X_\alpha$$

be the product space.

Let

$$x \in X$$

$$x = (X_\alpha : \alpha \in A)$$

and let P X be the component to which x belongs, $x \in P$.

We will show that an arbitrary point $y \in X$ belongs to P.

Let

$$F = X\{X_\alpha : \alpha \neq \alpha_1, \ldots, \alpha_m\} \times \bigcup_{\alpha_1} \times \ldots \times \bigcup_{\alpha_m}$$

be any basic open set containing $y \in X$. The set

$$G = X\{\{X_\alpha\} : \alpha \neq \alpha_1, \dots, \alpha_m\} \times X_{\alpha_1} \times \dots \times X_{\alpha_m}$$

is homeomorphic to

$$X_{\alpha_1} \times \dots \times X_{\alpha_m}$$

hence, it is connected.

Since $x \in G$

$$G \subset P$$

where P is the component of x.

The set

$$F \cap G \neq \phi$$

and $$P \cap P \neq \phi.$$

Thus,

$$y \in P = P.$$

X has one component; hence, it is connected.

● PROBLEM 17-36

Show that the space of real numbers R with the topology T generated by the closed-open intervals $[a, b) \subset R$ is totally disconnected.

SOLUTION:

DEFINITION OF TOTALLY DISCONNECTED SPACES

A topological space (X, T) is said to be totally disconnected if, for each

$$x, y \in X$$

open nonempty sets exist, U and V such that

$$X = U \cup V$$

$$U \cap V = \phi$$

and $$x \in U, \quad y \in V.$$

■

Let $x, y \in R$ and $x < y$. We define

$$U = (-\infty, y)$$

$$V = [y, +\infty).$$

Sets U and V are disjoint, open, nonempty sets and $R = U \cup V$. Also,

$$x \in U \quad \text{and} \quad y \in V.$$

Thus, R with the topology generated by the closed-open intervals is totally disconnected.

● PROBLEM 17-37

1. Show that the set Q of rational numbers with the absolute value topology is totally disconnected.

2. Show that a totally disconnected space is T_2 (i.e., Hausdorff).

SOLUTION:

1. Let $x, y \in Q$ and

$$x < y.$$

An irrational number a exists such that

$$x < a < y.$$

Define sets U and V

$$U = \{z \in Q : z < a\}$$

$$V = \{z \in Q : z > a\}.$$

Thus, $U \cup V$ is a disconnection of Q and

$$x \in U, y \in V.$$

Q is totally disconnected.

2. Let (X, T) be totally disconnected and let

$$x, y \in X.$$

Open, nonempty, disjoint sets U, V exist such that

$$x \in U \quad \text{and} \quad y \in V$$

$$U \cap V = \phi.$$

Hence, X is a Hausdorff space.

● PROBLEM 17-38

Prove that the components of a totally disconnected space are the singleton sets.

That statement leads to an equivalent definition of totally disconnected spaces. A space (X, T) is said to be totally disconnected if X is not connected and the only connected subspaces of X are the singleton subsets of X and ϕ.

SOLUTION:

Suppose $A \subset X$ is a component of X and

$$x, y \in A, \quad x \neq y.$$

Since X is totally disconnected, there are sets U and V such that

$$X = U \cup V, \quad U \cap V = \phi$$

$$x \in U, \quad y \in V.$$

U and V are open.

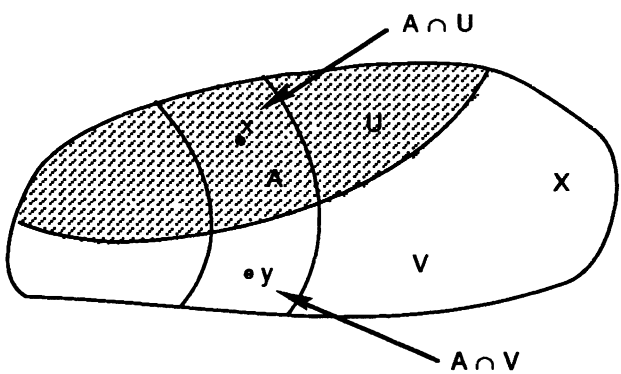


FIGURE 1

Hence, the sets

$$A \cap U \quad \text{and} \quad A \cap V$$

are nonempty, and $U \cup V$ is a disconnection of A. But as a component, A has to be connected.

Contradiction. Thus, A consists of one point.

● PROBLEM 17–39

Show that a space X with the discrete topology is totally disconnected and locally connected.

SOLUTION:

The only connected subsets of a discrete space are singleton sets and ϕ. Hence, X is totally disconnected.

DEFINITION OF LOCALLY CONNECTED SPACE

A space (X, T) is locally connected, if for any $x \in X$ and any neighborhood U of x, there is a connected neighborhood V of x such that

$$V \subset U.$$

■

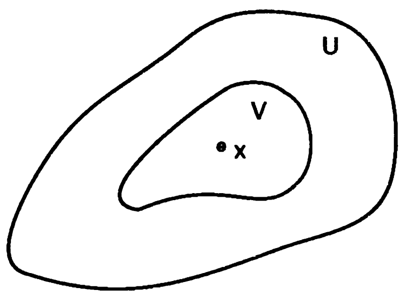


FIGURE 1

We can use this equivalent definition:

DEFINITION

Space (X, T) is said to be locally connected if there is an open neighborhood system for T such that for each $x \in X$, N_x consists of connected subspaces.

■

Let (X, T) be the space with discrete topology. For each

$$x \in X$$

we define

$$N_x = \{\{x\}\}$$

which establishes an open neighborhood system. Each subspace $\{x\}$ is connected.

● PROBLEM 17-40

Consider the space

$$X = \{(x, y) : 0 < x, y = \sin {}^1/_x\} \cup \{(0, 0)\}.$$

Show that X is not locally connected. Use this space to show that local connectedness is not preserved by a continuous function.

SOLUTION:

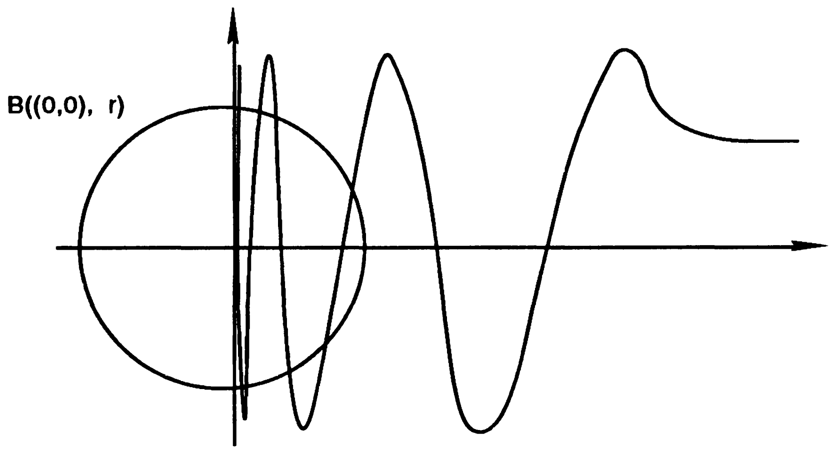


FIGURE 1

For any $0 < r < 1$, $B((0, 0), r)$ has an infinite number of components. X is connected but not locally connected.

On the other hand, X is a continuous image of a locally connected space.

Hence, the local connectedness is not generally perserved by continuous functions.

● PROBLEM 17-41

1. Let X and Y be locally connected. Show that $X \times Y$ is locally connected.

2. Show that a component of a locally connected space X is open.

SOLUTION:

1. Space X is locally connected if and only if X has a base B consisting of connected sets. Also, Y has a base B' consisting of connected sets. Since $X \times Y$ is a finite product, a base for $X \times Y$ is defined by

$$\{U \times V : U \in B, V \in B'\}.$$

Each of the sets $U \times V$ is connected, since both U and V are connected. The product $X \times Y$ has a base consisting of connected sets; hence, $X \times Y$ is locally connected.

2. Let A be a component of a locally connected space X and

$$a \in A.$$

Since X is locally connected, a belongs to at least one open connected set $U_a \subset A$.

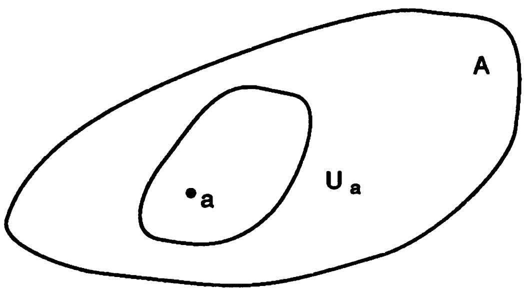


FIGURE 1

But A is a component of a; hence,

$$a \in U_a \subset A$$

and

$$A = \bigcup_{a \in A} U_a.$$

Thus A is open, as the union of open sets is.

● PROBLEM 17-42

Show that a compact, locally connected space (X, T) has a finite number of components.

SOLUTION:

Let

$$(A_\alpha : \alpha \in A\}$$

be the family of components of X. Hence, $\{A_\alpha\}$ forms an open cover of X.

Since X is compact, a finite subcover exists, say

$$A_{\alpha_1}, A_{\alpha_2}, \ldots, A_{\alpha_k}.$$

On the other hand, since $\{A_\alpha\}$ is the family of components, for $\alpha \neq \beta$ and α and $\beta \in A$, we have

$$A_\alpha \cap A_\beta = \phi.$$

Therefore, no A_α can be omitted from $\{A_\alpha\}$ and the remaining components still form a cover of X.

Hence,

$$A_{\alpha_1}, A_{\alpha_2}, \ldots, A_{\alpha_k}$$

are all of the components of X.

● PROBLEM 17-43

Continuous functions do not preserve local connectedness. But continuous open functions do. Prove:

THEOREM

If f is a continuous open function from a locally connected space (X, T) onto (Y, T').

$$f : X \to Y$$

then Y is locally connected.

■

SOLUTION:

f is a continuous open and onto function

$$f : \mathrm{X} \to Y.$$

Let U be any neighborhood of $y \in Y$.

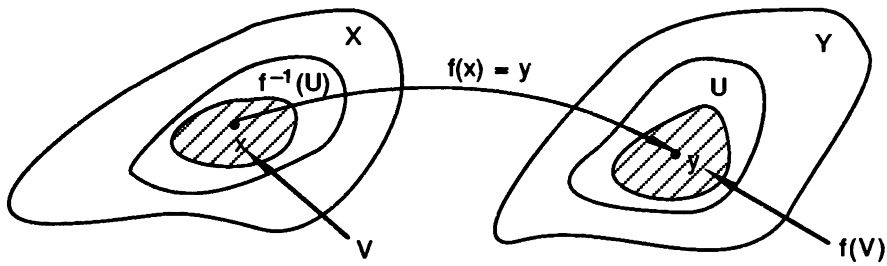


FIGURE 1

Since f is onto, $x \in X$ exists such that

$$f(x) = y.$$

f is continuous; hence, $f^{-1}(U)$ is an open set $x \in f^{-1}(U)$. But X is locally connected; thus, there is a connected neighborhood V of x such that

$$x \in V \subset f^{-1}(U).$$

Since f is continuous and open, $f(V)$ is a connected open set and

$$y \in f(V) \subset U.$$

Hence, Y is locally connected.

● PROBLEM 17–44

Prove the following:

THEOREM

If $\{X_\alpha : \alpha \in A\}$ is a family of connected locally connected spaces, then $\underset{\alpha}{\times} X_\alpha$ is locally connected. ■

SOLUTION:

Let

$$x = (x_\alpha) \in \underset{\alpha}{\times} X_\alpha$$

and let $U \subset \underset{\alpha}{\times} X_\alpha$ be the neighborhood of x. A member of the defining base exists

$$B = U_{\alpha_1} \times \dots \times U_{\alpha_k} \times \left(\underset{\alpha \neq \alpha_1, \dots, \alpha_k}{\times} X_\alpha \right)$$

such that

$$x \in B \subset U$$

and

$$x_{\alpha_l} \in U_{\alpha_l}.$$

Each coordinate space is locally connected; hence. connected open subsets V_{α_l} exist such that

$$V_{\alpha_l} \subset X_{\alpha_l}$$

and

$$x_{\alpha_l} \in V_{\alpha_l} \subset U_{\alpha_l}$$

for $l = 1, 2, \dots, k$.

Let

$$V = V_{\alpha_1} \times \dots \times V_{\alpha_k} \times \left(\underset{\alpha \neq \alpha_1, \dots, \alpha_k}{\times} X_\alpha \right).$$

V is connected because each X_α is connected and each V_{α_l} is connected. Since V is open and

$$x \in V \subset B \subset U,$$

X is locally connected.

● PROBLEM 17–45

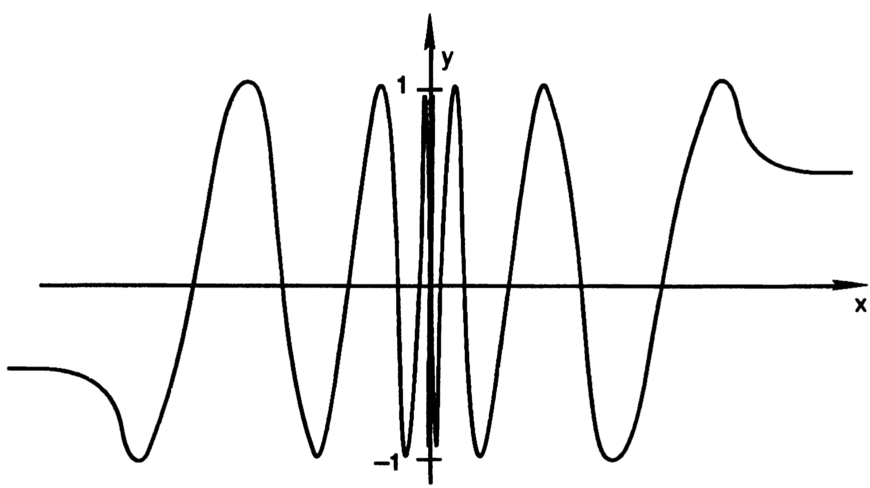


FIGURE 1

Which of the following sets is a continuum:

1. closed interval

2. closed n-dimensional cube

3. the set P of points defined by

$$y = \sin {}^1/_x \text{ for } 0 < |x| \le 1$$

$$|y| \le 1 \quad \text{for} \quad x = 0$$

SOLUTION:

A compact connected T_2–space (X, T) is called a continuum.

■

Note that any closed, bounded, and connected subset of R^n for any n is a continuum. This is due to the fact that the closure of any bounded subset of R^n is compact.

Also, any path in a T_2–space is a continuum.

We see that each of the spaces described in 1, 2, and 3 is a continuum.

● PROBLEM 17–46

Prove this:

THEOREM

The union of two continuua which have a common point is a continuum.

■

SOLUTION:

Let (X, T) and (Y, T') be compact connected spaces. We proved that a topological space which is the union of two compact sets is compact.

Hence, $X \cup Y$ is compact. We also proved the following:

If the sets A and B are connected and are not separated, then their union is connected.

Hence, $X \cup Y$ is connected.

Therefore, $X \cup Y$ is a continuum.

● PROBLEM 17-47

Let X and Y be continua such that

$$Y \mathrm{A} X$$

and let $X-Y$ be the union of two disjoint open sets A and B. Show that

$$Y \cup A \quad \text{and} \quad Y \cup B$$

are continua.

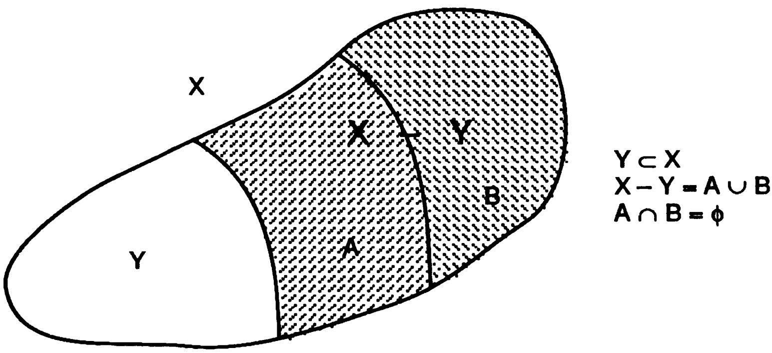


FIGURE 1

SOLUTION:

We shall apply the following:

THEOREM

If Y is a connected subset of the connected space X and

$$X-Y=A \cup B$$

where A and B are separated, then the sets

$$Y \cup A \quad \text{and} \quad Y \cup B$$

are connected.

Note that sets A and B are separated, if

$$\overline{A} \cap B = \phi \quad \text{and} \quad A \cap \overline{B} = \phi.$$

We conclude that $Y \cup A$ and $Y \cup B$ are continua.

Prove the following:

THEOREM

If $f : X \to Y$ is a homeomorphism and X is a continuum, then Y is a continuum.

■

SOLUTION:

We shall apply two theorems.

THEOREM

The continuous image of a compact set is compact.

■

THEOREM

The image under a continuous mapping of a connected space is a connected space.

■

Also, if $f : X \to Y$ is a homeomorphism and X is T_2, then Y is T_2.

Hence, if the spaces are homeomorphic and one is a continuum, then the other one is also a continuum.

The above proved theorem is a generalization of the theorem from analysis. Suppose X is a continuum and f is a homeomorphic real-valued function, then $f(X)$ is either a single point or a closed interval.

CHAPTER 18

METRIZABLE SPACES

• PROBLEM 18–1

Show that if X is metrizable, then $X \times X$ is also metrizable.

SOLUTION:

Suppose (X, T) is metrizable.

DEFINITION OF METRIZABLE SPACE

A space (X, T) is said to be metrizable if a metric d can be defined on X such that the topology induced by d is T. Otherwise, the space X is called non-metrizable.

■

Metric d exists such that d induces T. We define topology on $X \times X$ in the usual way. It can be proved that this topology is induced by the metric

$$d'((x_1, x_2), (y_1, y_2)) = d(x_1, y_1) + d(x_2, y_2).$$

Hence, if X is metrizable, so is $X \times X$.

• PROBLEM 18–2

R is the space of real numbers with the topology T generated by the basis consisting of open intervals. Show that (R, T) is metrizable.

SOLUTION:

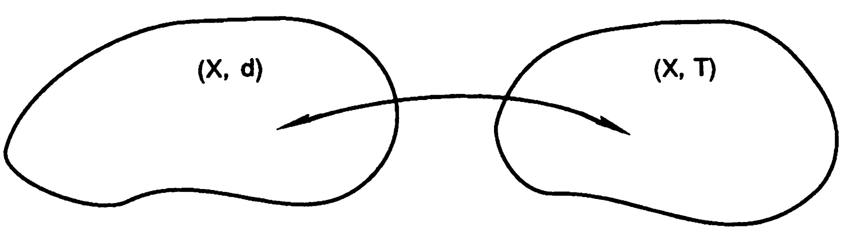


FIGURE 1

Suppose a metric space (X, d) is given. The metric d induces a certain topology T_d on X.

$$(X, d) \rightarrow (X, T_d)$$

In the metric space (X, d) a d-ball of radius r and center a is defined by

$$B(a, r) = \{x \in X : d(x, a) < r\}.$$

The family

$$\{B(a, r) : a \in X, r > 0\}$$

of all d-balls in X can serve as the basis for topology T.

Hence, we obtain a topology T on X determined (or induced) by the metric d.

Let R be the space of real numbers with the absolute value metric. Then this metric induces a topology with the open intervals as the basis. Hence, R with the topology generated by the basis consisting of open intervals is metrizable. The metric we are looking for is the absolute value metric.

● PROBLEM 18-3

Let $\{(X_n, d_n) : n \in N\}$ be a countable family of second countable metric spaces.

1. Show that the product space

$$\underset{n \in N}{\times} X_n$$

is second countable.

2. Show that

$$d(x, y) = \sum_{n \in N} \frac{\min(d_n(x_n, y_n), 1)}{2^n}.$$

is a metric on xX_n, where

$$x = (x_1, x_2, \ldots, x_n, \ldots);\ x_n, y_n \in X_n$$

$$y = (y_1, y_2, \ldots, y_n, \ldots)$$

SOLUTION:

1. Since each X_n is second countable, $\times X_n$ is second countable. We proved earlier that the product space of a countable family of nonempty spaces is second countable, if and only if each component space is second countable.

2. We have

$$d(x, y) = \sum_{n} \frac{\min(d_n(x_n, y_n), 1)}{2^n} \le \sum_{n} \frac{1}{2^n}.$$

Hence, $d(x, y)$ is defined for all

$$x, y \in \underset{n}{\times} X_n.$$

Also,

$$d(x, y) \geq 0$$

$d(x, y) = 0$ iff $x_n = y_n$ for all $n \in N$, hence, $x = y$.

$$d(x,y) = d(y,x)$$

$$d(x,y) = \sum_n \frac{\min(d_n(x_n, y_n), 1)}{2^n} \leq 1$$

$$\leq \sum_n \left[\frac{\min(d_n(x_n,z_n), 1)}{2^n} + \frac{\min(d_n(z_n, y_n), 1)}{2^n} \right] =$$

$$= d(x,z) + d(z,y).$$

Hence, d is a metric on $\underset{n}{\times} X_n$.

● PROBLEM 18-4

Using the metric defined in Problem 18–3, show that the space $\underset{n \in N}{\times} X_n$ is metrizable, where $\{(X_n, d_n) : n \in N\}$ is a countable family of second countable metric spaces.

SOLUTION:

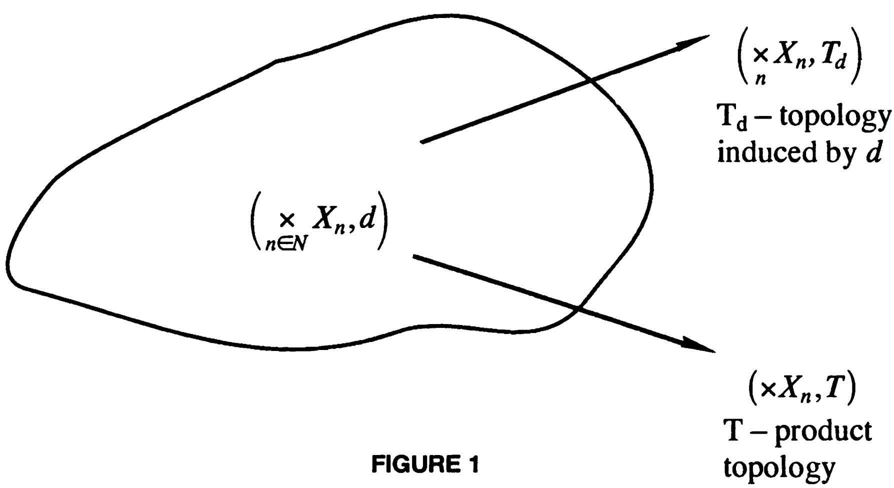


FIGURE 1

We have to show that

$$T_d = T.$$

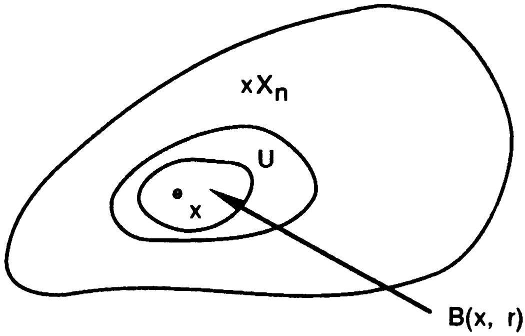


FIGURE 2

Let $x \in xX_n$ and let U be the basic neighborhood of x in the product topology. Then

$$U = \underset{n \in N}{\times} V_n$$

where V_n is open in X_n and for all but at most finitely many n, say

$$n_1, \dots, n_k$$

$$V_n = X_n$$

The sets $V_{n_1}, \dots, V_{n_k}$ are open in $X_{n_1}, \dots, X_{n_k}$ respectively. Hence, positive numbers $r_1, \dots, r_k$ exist such that

$$\begin{array}{ccccc} x_{n_1} & \in & B(x_{n_1}, r_1) & \subset & V_{n_1} \\ \vdots & & \vdots & & \vdots \\ x_{n_k} & \in & B(x_{n_k}, r_k) & \subset & V_{n_k} \end{array}$$

Choose

$$r = \min(r_1, \dots, r_k).$$

Then if $y \in B(x, r)$

$$y_n \in V_n$$

thus

$$y \in U \quad \text{and} \quad B(x, r) \subset U.$$

To complete the proof, we have to show that for any $B(x, r)$ there is U such that $U \subset B(x, r)$. Let $B(x, r)$ be a d-neighborhood of x. Choose $m \in N$ such that

$$\sum_{n=m}^{\infty} \frac{1}{2^n} < \frac{r}{2}$$

and let $r_1, \ldots, r_{m-1}$ be such that

$$r_1 + \ldots + r_{m-1} < {}^r/_2.$$

We define

$$U = \underset{n \in N}{x} B_n$$

where $B_n = B(x_n, r_n)$ for $n = 1, 2, \ldots, m-1$ and $B_n = X_n$ for other n's.

U is a basic neighborhood of x in the product topology and

$$U \subset B(x, r).$$

Hence, the product and metric topologies on xX_n are equivalent and $\underset{N}{\times} X_n$ is metrizable.

● PROBLEM 18–5

Explain why the Hilbert cube $\underset{n \in N}{\times} [0,1]$ is second countable and metrizable.

SOLUTION:

Since the closed interval [0, 1] is second countable, the Hilbert cube $\underset{N}{\times} [0,1]$ is second countable.

The closed interval [0, 1] with the absolute value topology is metrizable. By Problem 18–4 the product of the countable family of second countable metric spaces is metrizable.

● PROBLEM 18–6

Show that the sequence

$$a_n = \frac{1}{n}, \; n \in N$$

in the space of real numbers R with the absolute value metric is a Cauchy sequence.

SOLUTION:

We shall define Cauchy sequences.

DEFINITION OF CAUCHY SEQUENCES

A sequence (a_n) in (X, d) is a Cauchy sequence iff
$\forall\ \varepsilon > 0, \exists\ N_0 \in N$ such that

$$n, m > N_0 \Rightarrow d(a_n, a_m) < \varepsilon.$$

■

We take an arbitrary fixed $\varepsilon > 0$. The solution of the problem lies in finding $N_0 = N^0(\varepsilon)$ such that for every $n, m > N_0$

$$\left|\frac{1}{n} - \frac{1}{m}\right| < \varepsilon.$$

Let N_0 be such that

$$\frac{1}{N_0} < \frac{\varepsilon}{2}.$$

Note that we do not have to find the smallest possible N_0.

For any $n, m > N_0$

$$\frac{1}{n} < \frac{1}{N_0} \text{ and } \frac{1}{m} < \frac{1}{N_0}$$

and

$$\left|\frac{1}{n} - \frac{1}{m}\right| < \frac{1}{n} + \frac{1}{m} < \frac{1}{N_0} + \frac{1}{N_0} <$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence, $(^1/_n)$ is a Cauchy sequence in $(R, 1 \cdot 1)$.

● PROBLEM 18-7

Show that every convergent sequence in a metric space (X, d) is a Cauchy sequence.

SOLUTION:

Let (a_n) be a convergent sequence in (X, d)

$$a_n \underset{n \to \infty}{\longrightarrow} a.$$

In other words,

$$d(a_n, a) \to 0.$$

Then for every $\varepsilon > 0$ there exists N_0 such that

$$n > N_0 \Rightarrow d(a_n, a) < {}^{\varepsilon}/_{2}.$$

Hence, by the triangle inequality

$$n,m > N_0 \Rightarrow d(a_n, a_m) \leq d(a_n, a) + d(a_m, a) <$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus, (a_n) is a Cauchy sequence.

• PROBLEM 18-8

Find an example which illustrates that the converse of the statement in Problem 18–7 is not true; that is, not every Cauchy sequence is convergent.

SOLUTION:

Consider the space Q of rational numbers with the absolute value metric $(Q, 1 \cdot 1)$. The sequence is given as follows:

$$a_1 = 1$$

$$a_2 = 1.4$$

$$a_3 = 1.41$$

$$a_4 = 1.414$$

$$\vdots$$

Sequence (a_n) described above is the decimal expansion of $\sqrt{2}$. It is easy to verify that (a_n) is a Cauchy sequence. But (a_n) does not converge in Q (it converges in R). Hence, (a_n) is a sequence which is Cauchy but not convergent.

• PROBLEM 18-9

Show that every Cauchy sequence (a_n) in a metric space (X, d) is bounded and totally bounded.

SOLUTION:

Let (a_n) be a Cauchy sequence in (X, d). We shall prove that (a_n) is totally bounded (and thus bounded).

A subset A of a metric space (X, d) is totally bounded if A has an ε-net for every $\varepsilon > 0$. An ε-net for A is a finite set of points $\{a_1, \ldots, a_n\}$, such that for every $a \in A$ there is $a_k \in \{a_1, \ldots, a_n\}$ such that

$$d(a, a_k) < \varepsilon.$$

Let $\varepsilon > 0$. Since (a_n) is a Cauchy sequence,

$\exists\, N_0 \in N$ such that for every

$$n, m > N_0,\ d(a_n, a_m) < \varepsilon.$$

Hence, the diameter of the set

$$\{a_{N_0+1}, a_{N_0+2}, \ldots\}$$

is at most ε.

The ε-net for the sequence (a_n) is

$$\{a_1, a_2, \ldots, a_{N_0}, a_{N_0+1}\}.$$

● PROBLEM 18–10

Let (X, d) be a space with the trivial metric

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}.$$

Find a general formula for a Cauchy sequence in this space.

SOLUTION:

Suppose (a_n) is a Cauchy sequence in the trivial space.

Let $\varepsilon = {}^1/_2$. Then there exists N_0 such that

$$n, m > N_0 \Rightarrow d(a_n, a_m) < {}^1/_2 \text{ that is}$$

$$a_n = a_m.$$

Hence, a Cauchy sequence in the space with the trivial metric is of the form

$$(a_1, a_2, \ldots, a_k, a, a, a \ldots)$$

that is constant from a certain term on.

● PROBLEM 18-11

Let (a_n) be a Cauchy sequence in (X, d) and let (a_{K_n}) be its subsequence. Show that

$$d(a_n, a_{K_n}) \underset{n \to \infty}{\to} 0.$$

SOLUTION:

Let $\varepsilon > 0$ be an arbitrary positive number. Since (a_n) is a Cauchy sequence $\exists\, N_0 \in N$ such that,

$$n, m > N_0 - 1 \Rightarrow d(a_n, a_m) < \varepsilon.$$

Hence,

$$K_{N_0} \geq N_0 > N_0 - 1 \text{ and } d(a_{N_0}, a_{K_{N_0}}) < \varepsilon.$$

That is

$$\lim_{n \to \infty} d(a_n, a_{K_n}) = 0.$$

● PROBLEM 18-12

Let (a_n) be a sequence in a metric space (X, d). Let

$$A_1 = \{a_1, a_2, \ldots\}$$

$$A_2 = \{a_2, a_3, \ldots\}$$

$$A_3 = \{a_3, a_4, \ldots\}$$

$$\vdots$$

Show that

$$((a_n) \text{ is a Cauchy sequence}) \Leftrightarrow (\lim_{n \to \infty} \delta(A_n) = 0)$$

where $\delta(A_n)$ denotes the diameter of A_n.

SOLUTION:

$\Rightarrow$ Suppose (a_n) is a Cauchy sequence. For $\varepsilon > 0$ there exists $N_0 \in N$ such that for $n, m > N_0$

$$d(a_n, a_m) < \varepsilon.$$

Thus, for $n > N_0$

$$\delta(A_n) < \varepsilon.$$

Since ε was chosen arbitrarily,

$$\lim_{n\to\infty} \delta(A_n) = 0.$$

$\Leftarrow$

$$\lim_{n\to\infty} \delta(A_n) = 0.$$

For any $\varepsilon > 0$ there exists N_0 such that for any $n > N_0$

$$\delta(A_n) < \varepsilon.$$

In other words, for any $n, m > N_0$

$$d(a_n, a_m) < \varepsilon$$

and (a_n) there is a Cauchy sequence.

● PROBLEM 18–13

Let (a_n) be a Cauchy sequence in (X, d) and (b_n) be a sequence in (X, d) such that

$$d(a_n, b_n) < \frac{1}{n}$$

for every $n \in N$. Show that:

1. (b_n) is also a Cauchy sequence.
2. (b_n) converges to $a \in X$ iff (a_n) converges to $a \in X$.

SOLUTION:

1. Applying the triangle inequality,

$$d(b_m, b_n) \le d(b_m, a_m) + d(a_m, b_n) \le$$

$$\le d(b_m, a_m) + d(b_n, a_n) + d(a_n, a_m).$$

Let $\varepsilon > 0$. We can find $N_1 \in N$ such that for

$$n, m > N_1$$

we have

$$d(b_m, b_n) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + d(a_m, a_n).$$

Since (a_n) is a Cauchy sequence,

$\exists\, N_2 \in N$ such that

$$n, m > N_2 \Rightarrow d(a_m, a_n) \le \frac{\varepsilon}{3}.$$

Setting

$$N_3 = \max\{N_1, N_2\}$$

we get

$$n, m > N_3 \Rightarrow d(b_m, b_n) < \varepsilon.$$

Hence, (b_n) is a Cauchy sequence.

2. Similarly,

$$d(a_n, a) \le d(a_n, b_n) + d(b_n, a).$$

Hence,

$$\lim_{n \to \infty} d(a_n, a) \le \lim_{n \to \infty} d(a_n, b_n) + \lim_{n \to \infty} d(b_n, a).$$

But

$$\lim d(a_n, b_n) = 0.$$

If $b_n \to a$, then

$$\lim_{n \to \infty} d(a_n, a) \le \lim_{n \to \infty} d(b_n, a) = 0$$

and
$$\lim_{n \to \infty} a_n = a$$

Similarly, if $\quad a_n \underset{n \to \infty}{\to} a$ then $b_n \underset{n \to \infty}{\to} a.$

● PROBLEM 18–14

Show that if (x_n) is a Cauchy sequence in Euclidean m-space R^m

$$x_1 = (x_1^{(1)}, x_1^{(2)}, \dots, x_1^{(m)})$$

$$\vdots$$

$$x_n = (x_n^{(1)}, x_n^{(2)}, \dots, x_n^{(m)})$$

$$\vdots$$

then each of the sequences

$$\left(x_K^{(1)}\right), \left(x_K^{(2)}\right), \ldots, \left(x_K^{(m)}\right)$$

$k = 1, \ldots, n$ is a Cauchy sequence in R.

SOLUTION:

Let $\varepsilon > 0$. Since (x_n) is a Cauchy sequence,

$\exists\, N_0 \in N$ such that for $j, l > N_0$

$$d(x_j, x_l) = \sqrt{\left|x_j^{(1)} - x_l^{(1)}\right|^2 + \ldots + \left|x_j^{(m)} - x_l^{(m)}\right|^2} < \varepsilon.$$

Thus, for each term we obtain

$$\left|x_j^{(1)} - x_l^{(1)}\right| < \varepsilon$$

$$\vdots$$

$$\left|x_j^{(m)} - x_l^{(m)}\right| < \varepsilon.$$

Each of the m sequences

$$\left(x_k^{(1)}\right), \ldots, \left(x_k^{(m)}\right)$$

is a Cauchy sequence.

● PROBLEM 18-15

1. Show that the real line R with the usual metric is complete.
2. Give an example of a subspace of R which is not complete.

SOLUTION:

DEFINITION OF COMPLETE METRIC SPACE

A metric space (X, d) is complete if every Cauchy sequence (a_n) in X converges to a point $a \in X$.

$$\lim_{n\to\infty} a_n = a \in X.$$

The real line R is complete. This is the result of the Cauchy theorem: Every Cauchy sequence of real numbers converges to a real number.

2. Consider the open interval]0, 1[with the absolute metric. The sequence ($^1/_2$, $^1/_3$, $^1/_4$, $^1/_5$, ...) is a Cauchy sequence which converges, in the limit as

$$\lim_{n \to \infty} \frac{1}{n} = 0.$$

However, 0 does not belong to]0, 1[, hence,]0, 1[is not complete.

● PROBLEM 18-16

1. Is completeness a topological property?

2. Find the diameters of A, B, and C

$$A = \{(x, y) : x^2 + y^2 \le 1\} \subset R^2$$

$$B = \{(x, y) : x^2 + y^2 = 1\} \subset R^2$$

$$C = \{(x, y) : |x| < 1, |y| \le 2\} \subset R^2$$

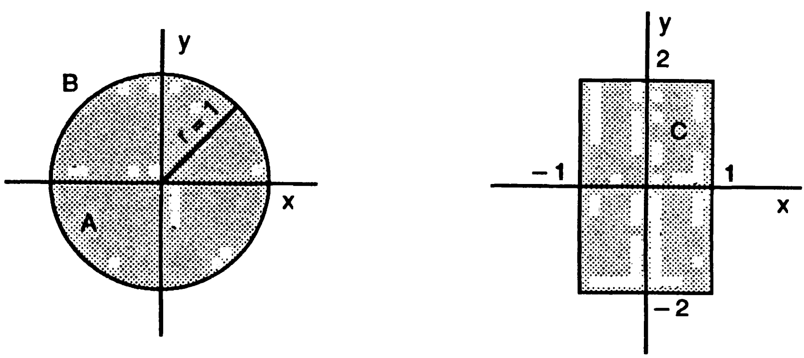


FIGURE 1

SOLUTION:

1. The real line R with the absolute metric is complete. As demonstrated in Problem 18-15, the open interval]0, 1[with the absolute metric is not complete. But R is homeomorphic to]0, 1[, then completeness is not a topological property.

2. The diameter of a subset A of a metric space (X, d) is defined to be the least upper bound of

$$\{d(x, y) : x, y \in A\}.$$

We denote the diameter of A by $\delta(A)$ or $d(A)$.

The diameter of A, B, and C are

$$\delta(A) = 1$$

$$\delta(B) = 1$$

$$\delta(C) = \sqrt{20}.$$

● PROBLEM 18–17

Show that for any subset A of a metric space (X, d)

$$\delta(A) = \delta(\overline{A}). \tag{1}$$

SOLUTION:

We shall prove that for any positive number $r > 0$

$$\delta(\overline{A}) < \delta(A) + r \tag{2}$$

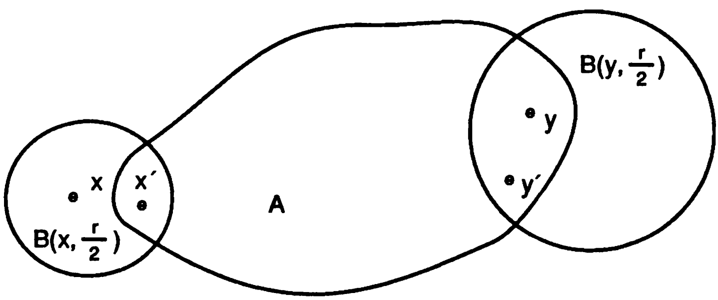


FIGURE 1

Suppose

$$x, y \in \overline{A}. \tag{3}$$

Then

$$B(x, {}^r/_2) \cap A \neq \phi$$

$$B(y, {}^r/_2) \cap A \neq \phi. \tag{4}$$

We choose x' and y' such that

$$x' \in B(x, {}^r/_2) \cap A$$

$$y' \in B(y, {}^r/_2) \cap A \tag{5}$$

Then

$$d(x, y) \le d(x, x') + d(y, x') \le$$

$$\le d(x, x') + d(y, y') + d(x', y') < r/2 + r/2 + \delta(A) =$$

$$= r + \delta(A). \tag{6}$$

Therefore,

$$\delta(\overline{A}) < \delta(A) + r \tag{7}$$

and

$$\delta(\overline{A}) \le \delta(A). \tag{8}$$

Since $A \subset \overline{A}$,

$$\delta(A) \le d(\overline{A}). \tag{9}$$

Thus,

$$\delta(A) = \delta(\overline{A}). \tag{10}$$

● PROBLEM 18–18

Prove the Cantor theorem.

CANTOR THEOREM

Let $\{A_n\}$, $n \in N$ be a countable family of closed nonempty subsets of complete space X such that

$$A_1 \supset A_2 \supset A_3 \ldots \text{ and } \delta(A_n) \underset{n \to \infty}{\to} 0 \tag{1}$$

Then

$$\bigcap_{n \in N} A_n \ne \phi. \tag{2}$$

■

SOLUTION:

For every $n \in N$ choose $a_n \in A_n$. We shall show that (a_n) is a Cauchy sequence. Let $\varepsilon > 0$. There exists

$$N_0 \in N$$

such that for $n > N_0$

$$\delta(A_n) < \varepsilon/2. \tag{3}$$

For $k, l > N_0$, $a_k, a_l \in A_{N_0+1}$.

Hence,

$$d(a_k, a_l) < \varepsilon. \tag{4}$$

Thus, (a_n) is a Cauchy sequence in (X, d). Since, by hypothesis, X is complete

$$\lim_{n \to \infty} a_n = a \in X. \tag{5}$$

Note that for any n

$$(a_n, a_{n+1}, \ldots)$$

is also a sequence which converges to a.

$$\{a_n, a_{n+1}, \ldots\} \subset A_n. \tag{6}$$

Since for each n, A_n is closed,

$$a \in \overline{A_n} = A_n. \tag{7}$$

Thus,

$$a \in \bigcap_{n \in N} A_n \text{ and } \bigcap_{n \in N} A_n \neq \phi. \tag{8}$$

• PROBLEM 18–19

Prove the following:

THEOREM

A metric space (X, d) is complete if and only if every nested sequence (i.e., $A_1 \supset A_2 \supset A_3 \ldots$) of nonempty closed sets such that $\delta(A_n) \to 0$ has a nonempty intersection, $\bigcap_n A_n \neq \phi$.

■

SOLUTION:

We have to show that

$$((X, d) \text{ complete}) \Leftrightarrow (\bigcap_n A_n \neq \phi).$$

For any countable family $\{A_n\}$ of closed, nonempty subsets of X such that

$$A_1 \supset A_2 \supset A_3 \ldots$$

and

$$\delta(A_n) \to 0.$$

In Problem 18–18 we proved $\Rightarrow$.

Now we shall show $\Leftarrow$.

Let $A_1 \supset A_2 \supset A_3 \ldots$ be any sequence of closed, nonempty subsets of X such that

$$\delta(A_n) \to 0, \cap A_n \neq \phi.$$

Let $\{X_n\}$, $n \in N$ be a Cauchy sequence in X. We define

$$B_1 = \{x_1, x_2, x_3, \ldots\}, A_1 = \overline{B}_1$$

$$B_2 = \{x_2, x_3, \ldots\}, A_2 = \overline{B}_2$$

$$B_3 = \{x_3, x_4, \ldots\}, A_3 = \overline{B}_3$$

$$\vdots$$

Then $A_1 \supset A_2 \supset A_3 \supset \ldots$ and $\delta(A_n) \to 0$ and $\{A_n\}$ are closed, nonempty subsets of X. Hence,

$$\bigcap_{n \in N} A_n \neq \phi.$$

Let

$$x \in \bigcap_n A_n.$$

We show that

$$x_n \to x \in X.$$

Let $\varepsilon > 0$. Then

$\exists\, N_0 \in N$ such that for $n > N_0$

$$\delta(B_n) < \varepsilon.$$

Hence (by Problem 18–17),

$$\delta(B_n) = \delta(A_n) < \varepsilon.$$

For all $n > N_0$

$$d(x_n, x) < \varepsilon.$$

Hence,

$$x_n \to x \in X.$$

● PROBLEM 18–20

> Show that any compact metric space (X, d) is complete.

SOLUTION:

A space X is said to be compact if given any open cover $\{U_i\}$, $i \in I$, of X, there is a finite subcover of $\{U_i\}$, $i \in I$.

Suppose $(A_n\}$, $n \in N$, are closed, nonempty subsets of X such that

$$A_1 \supset A_2 \supset A_3 \supset \ldots \tag{1}$$

We shall apply the following: X is compact if and only if for any family $\{F_i\}$, $i \in I$, of closed subsets of X such that the intersection of any finite number of the F_i is nonempty,

$$\bigcap_{i \in I} F_i \neq \phi. \tag{2}$$

Then

$$\bigcap_{n \in N} A_n \neq \phi. \tag{3}$$

Hence, by Problem 18–19, the space (X, d) is complete.

● PROBLEM 18–21

> 1. Show that a subspace of a complete metric space can be incomplete.
>
> 2. Show that any separable metric space (X, d) is homeomorphic to a dense subspace of a complete metric space.

SOLUTION:

1. The set of rational numbers forms an incomplete subspace of the complete set of real numbers.

2. Space X is said to be separable if X contains a countable dense subset.

Any separable metric space (X, d) has a metrizable compactification, say (Y, d'). If a metric space (Y, d') is compact, then (Y, d') is complete.

● PROBLEM 18-22

> Show that any closed subspace A of a complete metric space (X, d) is a complete metric space.

SOLUTION:

Suppose

$$(a_n),\ n \in N$$

is a Cauchy sequence in A. Then (a_n) is also a Cauchy sequence in X. Since X is complete

$$a_n \to a \in X.$$

But

$$a \in \overline{A} = A.$$

Hence, (a_n) converges in A and A is complete.

● PROBLEM 18-23

> Show that
>
> $$((\underset{n}{x} X_n, d) \text{ is complete}) \Leftrightarrow (\text{each } (X_n, d_n) \text{ is complete}).$$

SOLUTION:

Let us define d as follows:

$$d(x,y) = \sum_{n \in N} \frac{\min(d_n(x_n, y_n), 1)}{2^n}.$$

where

$$x = (x_1, x_2, x_3, \ldots)$$

$$y = (y_1, y_2, y_3, \ldots).$$

It can be shown that d is a metric in $\underset{n}{\times} X_n$.

$\Leftarrow$ Suppose each (X_n, d_n) is complete. Let

$$(a_k),\ k \in N$$

be a Cauchy sequence in $\underset{n}{x} X_n$. The l-th coordinate of a_k is denoted by

$$a_k(l).$$

The sequence $(a_k(l))$, $k \in N$ is a Cauchy sequence in X_l. Hence, $(a_k(l))$ converges in X_l for each $l \in N$. Thus, (a_k) converges in $\underset{n}{x} X_n$.

$\Rightarrow$ Suppose (X_1, d_1) is not complete. There is a Cauchy sequence

$$(a_k(1)),\ k \in N$$

in X_1, which does not converge. Choose from each X_n, $n > 1$ a point

$$x_n \in X_n.$$

We define a sequence as

$$(a_K(l)) = \begin{cases} a_K(1) & \text{for } l = 1 \\ x_l & \text{for } l \neq 1 \end{cases}$$

which is a Cauchy sequence but does not converge in $(\underset{n}{x} X_n, d)$.

Contradiction.

● PROBLEM 18–24

1. Show that

$$f: R^2 \rightarrow R^2$$

$$f(x) = \frac{x}{2}$$

defined on Euclidean R^2 space is a contracting mapping.

2. Show that a contracting mapping is continuous.

SOLUTION:

DEFINITION OF CONTRACTING MAPPING

A function $f: X \rightarrow X$, defined as (X, d) is called a contracting mapping if a real number $a \in R$, $0 \leq a \leq 1$ exists there, such that for every $x, y \in X$

$$d(f(x), f(y)) \leq a\, d(x, y) < d(x, y).$$

■

1. $$d(f(x), f(y)) = \|f(x) - f(y)\| = \left\|\frac{x}{2} - \frac{y}{2}\right\| =$$

$$= \frac{1}{2}\|x - y\| = \frac{1}{2}\, d(x, y).$$

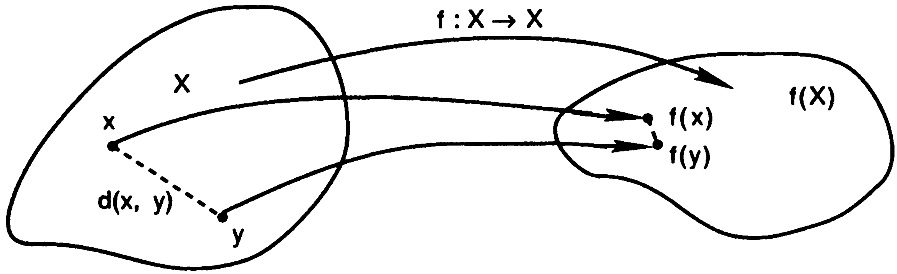


FIGURE 1

2. Let $f: X \to X$ be a contracting mapping on a metric space (X, d). A real number $0 \le a < 1$ exists such that for every $x, y \in X$

$$d(f(x), f(y)) \le a\, d(x, y).$$

Let $x_1 \in X$ and $\varepsilon > 0$.

We have

$$d(x, x_1) < \varepsilon \Rightarrow d(f(x), f(x_1)) \le a\, d(x, x_1) \le \varepsilon\, a < \varepsilon.$$

Hence, f is continuous.

● PROBLEM 18–25

Show that if $f: X \to X$ is a contracting mapping on a complete metric space (X, d) then there exists one and only one point $x_0 \in X$ such that

$$f(x_0) = x_0. \tag{1}$$

SOLUTION:

Let $a_0 \in X$. We define

$$f(a_0) = a_1, f(a_1) = a_2 = f^2(a_0), \ldots$$

$$f(a_{n-1}) = a_n = f^n(a_0), \ldots \tag{2}$$

and show that

$$a_0, a_1, a_2, \ldots$$

is a Cauchy sequence.

$$d(f^{k+l}(a_0), f^l(a_0)) \le a\, d(f^{k+l-1}(a_0), f^{l-1}(a_0)) \le \ldots$$

$$\ldots \le a^l\, d(f^k(a_0), \mathrm{a}_0) \le$$

$$\leq a^l[d(a_0, f(a_0)) + d(f(a_0), f^2(a_0)) + \ldots + d(f^{k-1}(a_0), f^k(a_0))]. \qquad (3)$$

But

$$d(f^{m+1}(a_0), f^m(a_0)) \leq a^m\, d(f(a_0), a_0). \qquad (4)$$

Hence, from (3) and (4) we obtain

$$d(f^{k+l}(a_0), f^l(a_0)) \leq a^l\, d(f(a_0), a_0)\,[1 + a + a^2 + \ldots + a^{k-1}]. \qquad (5)$$

Since $a < 1$,

$$1 + a + a^2 + \ldots + a^{k-1} \leq \frac{1}{1-a}. \qquad (6)$$

From (5) and (6) we find

$$d(f^{k+l}(a_0), f^l(a_0)) \leq \frac{a^l}{1-a}\, d(f(a_0), a_0). \qquad (7)$$

For $\varepsilon > 0$, we define

$$\lambda = \begin{cases} \varepsilon(1-a) & \text{if} \quad d(f(a_0), a_0) = 0 \\ \dfrac{\varepsilon(1-a)}{d(f(a_0), a_0)} & \text{if} \quad d(f(a_0), a_0) \neq 0 \end{cases} \qquad (8)$$

Since $a < 1$, $M \in N$ exists such that

$$a^M < \lambda. \qquad (9)$$

For $k \geq l > M$,

$$d(a_k, a_l) \leq a^l \cdot \frac{1}{1-a} \cdot d(f(a_0), a_0) < \frac{\lambda}{1-a} \cdot d(f(a_0), a_0) \leq \varepsilon. \qquad (10)$$

Thus, $(a_1, a_2, a_3, \ldots)$ is a Cauchy sequence and since X is complete $x_0 \in X$ exists such that

$$\lim_{n \to \infty} a_n = x_0. \qquad (11)$$

Function f is continuous, thus

$$f(x_0) = f(\lim a_n) = \lim f(a_n) = \lim a_n = x_0. \qquad (12)$$

Point x_0 is unique. For, suppose $f(x_0) = x_0$ and $f(y_0) = y_0$, then

$$d(x_0, y_0) = d(f(x_0), f(y_0)) \leq a\, d(x_0, y_0). \qquad (13)$$

But $a < 1$. Therefore,

$$d(x_0, y_0) = 0$$

and $x_0 = y_0$.

● PROBLEM 18-26

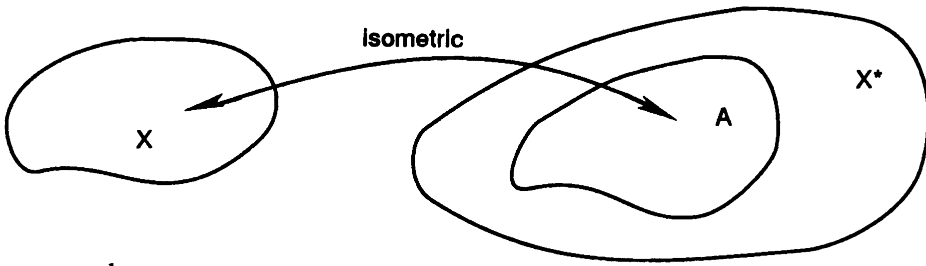
Show that the set of real numbers R is a completion of the set of rational numbers Q.

SOLUTION:

DEFINITION OF COMPLETION OF METRIC SPACE

A metric space X^* is called a completion of a metric space X if X^* is complete and X is isometric to a dense subset of X^*.

■



X^* – complete.
X and A are isometric.
$\overline{A} = X^*$.

FIGURE 1

Consider the set Q of rational numbers. The closure of Q is the entire set R of real numbers, i.e.,

$$\overline{Q} = R.$$

The set R is complete. Hence, the set R of real numbers is a completion of the set Q of rational numbers.

● PROBLEM 18-27

Let $C[X]$ denote the collection of all Cauchy sequences on X. The relation $\sim$ on $C[X]$ is defined by

$$((a_n) \sim (b_n)) \Leftrightarrow (\lim_{n \to \infty} d(a_n, b_n) = 0). \qquad (1)$$

Show that $\sim$ is an equivalence relation on $C[X]$.

SOLUTION:

$$(a_n) \sim (a_n)$$

is obvious, since $d(x, x) = 0$

$$((a_n) \sim (b_n)) \Rightarrow ((b_n) \sim (a_n))$$

because

$$d(x, y) = d(y, x).$$

Suppose $(a_n) \sim (b_n)$ and $(b_n) \sim (c_n)$. Then

$$\lim d(a_n, b_n) = \lim d(b_n, c_n) = 0.$$

From the definition of a metric,

$$d(a_n, c_n) \leq d(a_n, b_n) + d(b_n, c_n).$$

Taking the limit

$$\lim d(a_n, c_n) \leq \lim d(a_n, b_n) + \lim d(b_n, c_n) = 0.$$

Thus,

$$(a_n) \sim (c_n).$$

The relation $\sim$ is an equivalence relation on $C[X]$.

● PROBLEM 18–28

Show that

$$((a_n) \sim (b_n)) \Leftrightarrow \begin{pmatrix} (a_n) \text{ and } (b_n) \text{ are subsequences} \\ \text{of some Cauchy sequence } (c_n) \end{pmatrix}.$$

SOLUTION:

$\Leftarrow$ A Cauchy sequence (c_n) exists such that

$$(a_n) = (c_{k_n}) \text{ and } (b_n) = (c_{l_n}).$$

Then

$$\lim_{n \to \infty} d(a_n, b_n) = \lim_{n \to \infty} d(c_{k_n}, c_{l_n}) = 0$$

since (c_n) is a Cauchy sequence.

$\Rightarrow$ Suppose

$$(a_n) \sim (b_n)$$

that is

$$\lim_{n \to \infty} d(a_n, b_n) = 0.$$

We define a sequence

$$(c_n) = (a_1, b_1, a_2, b_2, a_3, b_3, \ldots)$$

that is

$$(c_n) = \begin{cases} a_{\frac{n+1}{2}} & \text{if } n \text{ is odd} \\ b_{\frac{n}{2}} & \text{if } n \text{ is even} \end{cases}.$$

and show that (c_n) is a Cauchy sequence. Let $\varepsilon > 0$, then $\exists\, n_1 \in N$ such that

$$n > n_1 \Rightarrow d(a_n, b_n) < {}^{\varepsilon}/_2$$

$\exists\, n_2 \in N$ such that

$$m, n > n_2 \Rightarrow d(a_m, a_n) < {}^{\varepsilon}/_2$$

$\exists\, n_3 \in N$ such that

$$m, n > n_3 \Rightarrow d(b_m, b_n) < {}^{\varepsilon}/_2$$

Set $n_0 = \max(n_1, n_2, n_3)$. Then for $m > 2n_0$

$$\frac{1}{2} m > n_1, \frac{1}{2} m > n_3$$

$$\frac{m+1}{2} > n_1, \frac{m+1}{2} > n_2.$$

We shall show that

$$m, n > 2n_0 \Rightarrow d(c_m, c_n) < \varepsilon.$$

We have

$$m, n \text{ even} \Rightarrow c_m = b_{\frac{m}{2}}, c_n = b_{\frac{n}{2}}$$

then

$$d(c_m, c_n) < {}^{\varepsilon}/_2 < \varepsilon$$

$$m, n \text{ odd} \Rightarrow c_m = a_{\frac{m+1}{2}}, c_n = a_{\frac{n+1}{2}}$$

then

$$d(c_m, c_n) < {}^{\varepsilon}/_2 < \varepsilon$$

$$m,n \text{ one odd and one even} \Rightarrow c_m = b_{\frac{m}{2}},\ c_n = a_{\frac{n+1}{2}}$$

then

$$d(c_m,c_n) \le d(a_{\frac{m}{2}},b_{\frac{m}{2}}) + d(a_{\frac{m}{2}},a_{\frac{n+1}{2}}) < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Hence, (c_n) is a Cauchy sequence.

● PROBLEM 18–29

Describe the construction of a completion X^* of a metric space (X, d).

SOLUTION:

A metric space (X, d) is given. By $C[X]$ we denote the collection of all Cauchy sequences in X. Among Cauchy sequences the relation $\sim$ is introduced as follows:

$$(a_n) \sim (b_n) \text{ iff } \lim_{n \to \infty} d(a_n,b_n) = 0.$$

In Problem 18–27 we showed that $\sim$ is an equivalence relation. We define X^*

$$X^* = \frac{C[X]}{\sim}$$

as the quotient space. Elements of X^* are the equivalence classes $[(a_n)]$ of Cauchy sequences (a_n) of $C[X]$. Thus, if $(a_n) \in [(a_n)]$ and $(b_n) \in [(a_n)]$ we have $(a_n) \sim (b_n)$ and $\lim_{n \to \infty} d(a_n,b_n) = 0$.

Later we shall show that the function δ defined by

$$\delta([(a_n)], [(b_n)]) = \lim_{n \to \infty} d(a_n,b_n)$$

is a metric on X^*.

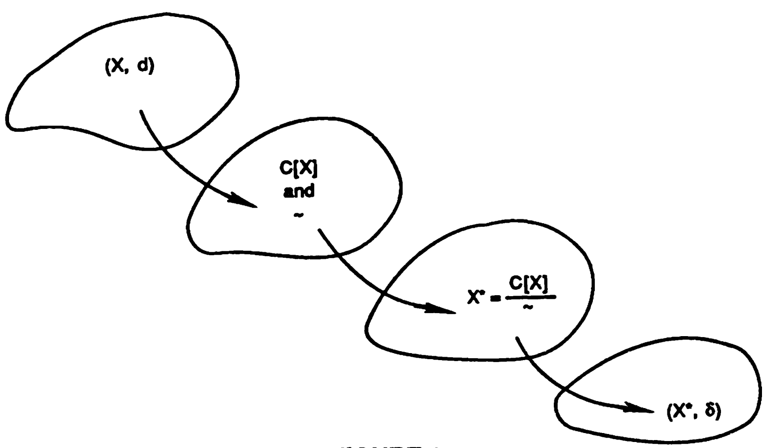


FIGURE 1

Show that the function δ defined in Problem 18–29 by

$$\delta([(a_n)], [(b_n)]) = \lim_{n \to \infty} d(a_n, b_n)$$

is well-defined, i.e., that

$$\begin{pmatrix} (a'_n) & \sim & (a_n) \\ & & \\ (b'_n) & \sim & (b_n) \end{pmatrix} \Rightarrow (\lim d(a'_n, b'_n) = \lim d(a_n, b_n).$$

SOLUTION:

Suppose $(a'_n) \sim (a_n)$ and $(b'_n) \sim (b_n)$ then

$$\lim_{n \to \infty} d(a'_n, a_n) = 0$$

and

$$\lim_{n \to \infty} d(b'_n, b_n) = 0.$$

Set

$$a = \lim d(a_n, b_n)$$

$$a' = \lim d(a'_n, b'_n).$$

We have

$$d(a_n, b_n) \le d(a_n, a'_n) + d(b_n, b'_n) + d(a'_n, b'_n).$$

Let $\varepsilon > 0$, then

$\exists\, n_1 \in N$ such that

$$n > n_1 \Rightarrow d(a_n, a'_n) < \varepsilon/3$$

$\exists\, n_2 \in N$ such that

$$n > n_2, \Rightarrow d(b_n, b'_n) < \varepsilon/3$$

$\exists\, n_3 \in N$ such that

$$n > n_3 \Rightarrow d(a'_n, b'_n) - a' < \varepsilon/3$$

For

$$n > \max(n_1, n_2, n_3)$$

we get

$$d(a_n,b_n) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + a' = a' + \varepsilon$$

and

$$\lim_{n\to\infty} d(a_n,b_n) = a \le a' + \varepsilon.$$

Hence, since $\varepsilon > 0$ is arbitrary

$$a \le a'.$$

In the same way we show that

$$a' \le a$$

and conclude that

$$a' = a.$$

● PROBLEM 18–31

Show that

$$\delta([(a_n)], [(b_n)]) = \lim_{n\to\infty} d(a_n,b_n)$$

is metric on

$$X^* = \frac{C[X]}{\sim}$$

(compare Problem 18–29).

SOLUTION:

We have already shown that δ is well-defined (Problem 18–30).

1. $$\delta([a_n)], [(b_n)] \ge 0$$

because $d(a_n, b_n) \ge 0$.

$$\delta([a_n)], [(a_n)] = 0$$

because $d(a_n, b_n) = 0$.

2. $$\delta([(a_n)], [(b_n)]) = \delta([(b_n)], [(a_n)])$$

because $d(a_n, b_n) = d(b_n, a_n)$.

3. $$\delta([(a_n)], [(b_n)]) = \lim_{n\to\infty} d(a_n,b_n) \le$$

$$\leq \lim_{n\to\infty} d(a_n, c_n) + \lim_{n\to\infty} d(b_n, c_n) =$$

$$= \delta([(a_n)], [(c_n)]) + \delta([(b_n)], [(c_n)]).$$

4. If $[(a_n)] \neq [(b_n)]$ then

$$\delta([(a_n)], [(b_n)]) > 0.$$

Indeed

$$\delta([(a_n)], [(b_n)]) = \lim_{n\to\infty} d(a_n, b_n) > 0$$

because

$$(a_n) \text{ is not in relation} \sim \text{with } (b_n) \Rightarrow \lim d(a_n, b_n) > 0.$$

Hence,

$$\delta([(a_n)], [(b_n)]) > 0.$$

● PROBLEM 18-32

Suppose $a \in X$ then $(a, a, a, \ldots)$ is a Cauchy sequence

$$(a, a, a, \ldots) \in C[X].$$

We define space $\hat{X}$ as follows:

$$\hat{a} \in \hat{X} = \{\hat{a} : a \in X\}$$

$\hat{a} = [(a, a, a, \ldots)]$. Space $\hat{X}$ is a subspace of X^*.

Now, suppose $(a_n) \in C[X]$. Show that

$$a = [(a_n)] \in X^*$$

is the limit of the sequence

$$(\hat{a}_1, \hat{a}_2, \hat{a}_3, \ldots)$$

in $\hat{X}$.

SOLUTION:

Sequence (a_n) is a Cauchy sequence in X. We have

$$\lim_{n\to\infty} \delta(a, \hat{a}_n) = \lim_{n\to\infty} (\lim_{n\to\infty} d(a_m, a_n)) =$$

$$= \lim_{\substack{n\to\infty \\ m\to\infty}} d(a_m, a_n) = 0.$$

Thus,

$$(\hat{a}_n) \to a.$$

● PROBLEM 18–33

1. Show that X is isometric to space $\hat{X}$ defined in Problem 18–32.

2. Show that $\hat{X}$ is dense in X^*.

SOLUTION:

1. Suppose $a, b \in X$. Then

$$\hat{a} = [(a, a, a, \ldots)] \in \hat{X}$$

$$\hat{b} = [(b, b, b, \ldots)] \in \hat{X}$$

and

$$\delta(\hat{a}, \hat{b}) = \lim_{n \to \infty} d(a, b) = d(a, b).$$

Hence, X is isometric to $\hat{X}$.

2. We shall show that every point in X^* is the limit of a sequence in $\hat{X}$. Let

$$a = [(a_1, a_2, a_3, \ldots)] \in X^*.$$

Then $(a_1, a_2, a_3, \ldots)$ is a Cauchy sequence in X. Hence, a is the limit of the sequence

$$(\hat{a}_1, \hat{a}_2, \hat{a}_3, \ldots)$$

in $\hat{X}$. Thus, $\hat{X}$ is dense in X^*.

● PROBLEM 18–34

Show that (X^*, d) is a completion of X, i.e., that every Cauchy sequence in (X^*, δ) converges.

SOLUTION:

Let

$$(a_1, a_2, a_3, \ldots)$$

be a Cauchy sequence in (X^*, δ). In Problem 18–33 we proved that

$$\hat{X} \text{ is dense in } X^*.$$

Hence,

$\forall\, n \in N \quad \exists\, \hat{a}_n \in \hat{X}$ such that

$$\delta(\hat{a}_n, a_n) < \frac{1}{n}.$$

Hence, $(\hat{a}_1, \hat{a}_2, \hat{a}_3, \ldots)$ is also a Cauchy sequence.

By Problem 18–33 $(\hat{a}_1, \hat{a}_2, \hat{a}_3, \ldots)$ converges to

$$a = [(a_1, a_2, \ldots)] \in X^*.$$

Thus, (a_n) also converges to a. We conclude that (X^*, δ) is complete.

● PROBLEM 18–35

Show that

$$(Y^* \text{ is a completion of } X) \Rightarrow (Y^* \text{ is isometric to } X^*).$$

SOLUTION:

X is a subspace of Y^*. Thus for every

$$y \in Y^*$$

there exists a Cauchy sequence $(y_1, y_2, y_3, \ldots)$ in X converging to y. Define a mapping

$$f: Y^* \to X^*$$

by

$$f(y) = [(y_1, y_2, \ldots)].$$

We shall show that f is an isometry between Y^* and X^*. Mapping f is well-defined. Indeed, suppose

$$(y'_1, y'_2, y'_3, \ldots) \to y$$

where (y'_n) is a sequence in X.

Then

$$\lim_{n \to \infty} d(y_n, y'_n) = 0$$

and

$$[(y_n)] = [(y'_n)].$$

Also f is subjective. Suppose

$$[(x_1, x_2, \ldots)] \in X^*.$$

Then

$$(x_1, x_2, \ldots)$$

is a Cauchy sequence in $X \subset Y^*$. But Y^* is complete. Hence, (x_n) converges to $x \in Y^*$ and

$$f(x) = [(x_n)].$$

Now suppose $x, y \in Y^*$. Then there are sequences (x_n) and (y_n) in X such that

$$x_n \rightarrow x$$

$$y_n \rightarrow y.$$

We have

$$\delta(f(x), f(y)) = \delta([(x_n)], [(y_n)]) =$$

$$= \lim_{n \to \infty} d(x_n, y_n) = d(x, y).$$

Hence, f is an isometry between Y^* and X^*.

● PROBLEM 18-36

1. Show that the set Z of integers is a nowhere dense subset of the real line R.

2. Show that the set Q of rational numbers is not nowhere dense in R.

SOLUTION:

1. A subset A of a topological space (X, T) is nowhere dense in X iff the interior of the closure of A is empty

$$int(\overline{A}) = \phi.$$

The set Z is closed and its interior is empty. Hence,

$$int(\overline{Z}) = int(Z) = \phi.$$

2. The closure of Q is R

$$\overline{Q} = R$$

and

$$int(\overline{Q}) = int(R) = R \neq \phi.$$

Thus, the set Q of rational numbers is not nowhere dense in R.

● PROBLEM 18-37

Let U be an open subset of the metric space (X, d) and let M be nowhere dense in X. Show that $x \in X$ exists and $\varepsilon > 0$ such that

$$B(x, \varepsilon) \subset U$$

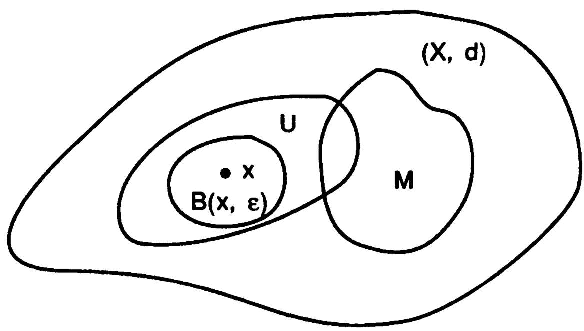


FIGURE 1

SOLUTION:

Let us denote

$$U \cap \overline{M^c} = V.$$

Then

$$V \subset U \quad \text{and} \quad V \cap M = \phi.$$

Set V is nonempty because U is open and $\overline{M^c}$ is dense in X.

$$V \neq \phi.$$

Hence,

$$x \in V.$$

V is open because U is open and $\overline{M^c}$ is open.

Thus, $\varepsilon > 0$ exists such that

$$B(x, \varepsilon) \subset V$$

and

$$B(x, \varepsilon) \cap M = \phi.$$

● PROBLEM 18-38

Show that if M is a nowhere dense subset of X then $\overline{M^c}$ is dense in X. (We denote $M^c = X - M$.)

SOLUTION:

Suppose on the contrary that $\overline{M^c}$ is not dense in X. Then $x \in X$ exists and an open set U such that

$$x \in U$$

$$U \cap \overline{M^c} = \phi.$$

Hence,

$$x \in U \subset \overline{M}$$

and

$$x \in int(\overline{M}).$$

It is a contradiction, because M is nowhere dense in X, $int(\overline{M}) = \phi$. Thus, $\overline{M^c}$ is dense in X.

● PROBLEM 18-39

Prove Baire's Category Theorem.

THEOREM

Every complete metric space (X, d) is of second category. ■

SOLUTION:

DEFINITION OF FIRST CATEGORY SPACES

A topological space (X, T) is said to be of first category if X is the countable union of nowhere dense subsets of X. All other spaces are of second category.

■

Let $A \subset X$ be of first category. We will show that $A \neq X$. Since A is of first category, by definition

$$A = M_1 \cup M_2 \cup \dots \tag{1}$$

where each M_n is a nowhere dense subset of X.

Since M_1 is nowhere dense in X, there exists $x_1 \in X$ and $\varepsilon_1 > 0$ such that

$$B(x_1, \varepsilon_1) \cap M_1 = \phi. \tag{2}$$

Then

$$\overline{B(x_1, \tfrac{\varepsilon_1}{2})} \cap M_1 = \phi. \tag{3}$$

Since $B(x_1, \frac{\varepsilon_1}{2})$ is open and M_2 is nowhere dense in X, by Problem 18–38 we conclude that

$\exists\, x_2 \in X, \exists\, \varepsilon_2 > 0$ such that

$$B(x_2, \varepsilon_2) \subset B(x_1, \tfrac{\varepsilon_1}{2}) \subset \overline{B(x_1, \tfrac{\varepsilon_1}{2})} \tag{4}$$

and

$$B(x_2, \varepsilon_2) \cap M_2 = \phi. \tag{5}$$

Set

$$\varepsilon_2 \le \frac{\varepsilon_1}{2}. \tag{6}$$

Then

$$\overline{B(x_2, \tfrac{\varepsilon_2}{2})} \subset \overline{B(x_1, \tfrac{\varepsilon_1}{2})} \tag{7}$$

and

$$\overline{B(x_2, \tfrac{\varepsilon_2}{2})} \cap M_2 = \phi. \tag{8}$$

In this way we get a nested sequence of closed sets

$$\overline{B(x_1, \tfrac{\varepsilon_1}{2})} \supset \overline{B(x_2, \tfrac{\varepsilon_2}{2})} \supset \overline{B(x_3, \tfrac{\varepsilon_3}{2})} \supset \dots \tag{9}$$

such that for every $n \in N$

$$\overline{B(x_n, \frac{\varepsilon_n}{2})} \cap M_n = \phi \tag{10}$$

and
$$\frac{\varepsilon_n}{2} \leq \frac{\varepsilon_1}{2^n}.$$

Hence,
$$\lim_{n \to \infty} \varepsilon_n \leq \lim_{n \to \infty} \frac{\varepsilon_1}{2^{n-1}} = 0$$

and $x \in X$ exists such that
$$x \in \bigcap_{n-1}^{\infty} \overline{B(x_n, \frac{\varepsilon_n}{2})}. \tag{11}$$

Also, $x \notin M_n$ for every $n \in N$ and
$$x \notin A.$$

Thus, (X, d) is of second category.

CHAPTER 19

ELEMENTS OF HOMOTOPY THEORY

Show that the two continuous functions,

$$f: X \to R$$

$$g: X \to R$$

where R is the space of real numbers with standard topology, are always homotopic.

SOLUTION:

DEFINITION OF HOMOTOPIC FUNCTIONS

Two continuous functions

$$f: X \to Y$$

$$g: X \to Y$$

are given. Functions f and g are said to be homotopic, if a continuous function h

$$h: X \times [0, 1] \to Y$$

exists, such that

$$h(x, t) \in Y$$

$$h(x, 0) = f(x) \quad \text{and} \quad h(x, 1) = g(x).$$

■

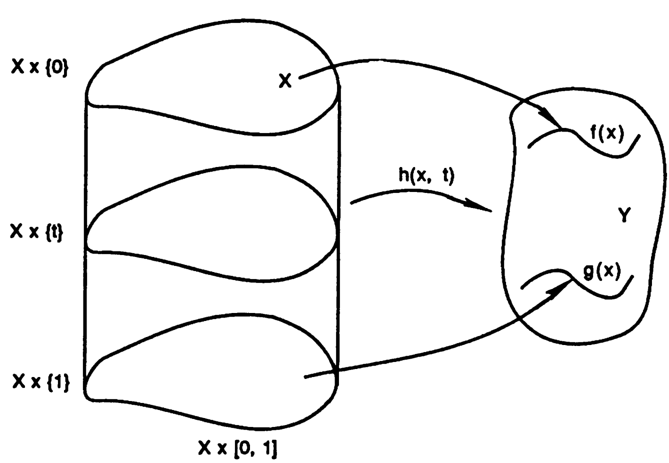


FIGURE 1

If $Y = R$ (or more generally, $Y = R^n$), then the functions f and g are always homotopic. Indeed,

$$h(x, t) = f(x) + t[g(x) - f(x)]$$

and $h(x, t)$ is continuous.

● PROBLEM 19-2

> Show that the identity function on the unit disk is homotopic to a constant function which maps the whole disk into (0, 0).

SOLUTION:

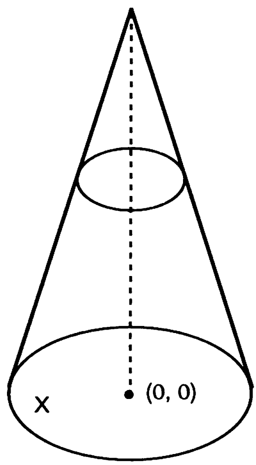


FIGURE 1

Suppose X and Y are unit disks and f is the identity function

$$f(x) = x \quad \text{for} \quad x \in X.$$

Let $g(x)$ denote a constant function defined on the unit disk

$$g(x) = (0, 0) \quad \text{for} \quad x \in X.$$

We shall define function $h(x, t)$.

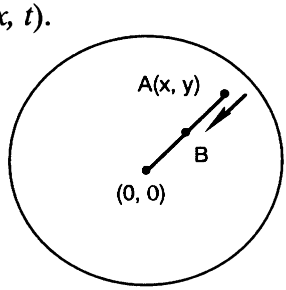


FIGURE 2

Let $A = (x, y)$ represent a point in the disk. Point $h(A, t) = B$ is the point on $\overline{(0, 0)\ (x, y)}$ that is t times the distance from (x, y) to $(0, 0)$. Hence

$$h(A, 0) = A$$

$$h(A, 1) = (0, 0) = \text{const.}$$

● PROBLEM 19-3

> Let Y^X denote the set of all continuous functions from X into Y, and let $f \sim g$ dneote that f is homotopic to g. Show that if $f \in Y^X$, then
>
> $$f \sim f.$$

SOLUTION:

The homotopy

$$h : X \times [0, 1] \rightarrow Y$$

between f and g is given by

$$h(x, t) = f(x)$$

for all $x \in X$ and all $t \in [0, 1]$.

We must show that $h(x, t)$ is continuous and that

$$h(x, 0) = h(x, 1) = f(x).$$

Suppose U is an open subset of Y. Then

$$h^{-1}(U) = f^{-1}(U) \times [0, 1].$$

Since f is continuous $f^{-1}(U)$ is an open subset of X and $f^{-1}(U) \times [0, 1]$ is an open subset of $X \times [0, 1]$. Hence, $h(x, t)$ is continuous.

● PROBLEM 19-4

> Show that if $f \sim g$, then $g \sim f$.

SOLUTION:

Since $f \sim g$, there is a homotopy

$$h : X \times [0, 1] \rightarrow Y$$

such that

$$h(x, 1) = f \quad \text{and} \quad h(x, 0) = g.$$

Let us define H by

$$H : X \times [0, 1] \to Y$$

$$H(x, t) = h(x, 1 - t)$$

for all $x \in X$ and all $t \in [0, 1]$.

We have

$$H(x, 0) = h(x, 1) = f$$

$$H(x, 1) = h(x, 0) = g.$$

Since $h(x, t)$ is continuous, so is $H(x, t)$.

Hence, $H(x, t)$ is a homotopy between g and f.

● PROBLEM 19-5

Show that the relation, $f \sim g$

$$f \text{ is homotopic to } g$$

is an equivalence relation.

SOLUTION

In Problem 19–3, we have shown that

$$f \sim f$$

and in Problem 19–4, we have shown that

$$\text{if } f \sim g, \text{ then } g \sim f.$$

It remains to be shown that, if $f \sim g$ and $g \sim k$, then $f \sim k$. Since $f \sim g$, there is a homotopy

$$h_1 : X \times [0, 1] \to Y$$

such that

$$h_1(x, 0) = g(x) \quad \text{and}$$

$$h_1(x, 1) = f(x) \text{ for all } x \in X.$$

Similarly, since $g \sim k$, there is a homotopy

$$h_2 : X \times [0, 1] \to Y$$

such that

$$h_2(x, 0) = k(x) \quad \text{and}$$

$$h_2(x, 1) = g(x) \text{ for all } x \in X.$$

Let us define mapping

$$h : X \times [0, 1] \to Y$$

by

$$h(x,t) = \begin{cases} h_2(x, 2t) & \text{for} \quad 0 \le t \le \frac{1}{2} \\ h_1(x, 2t-1) & \text{for} \quad \frac{1}{2} \le t \le 1 \end{cases}$$

for all $x \in X$.

Note that

$$h(x, 0) = h_2(x, 0) = k(x)$$

$$h(x, 1) = h_1(x, 1) = f(x)$$

$$h(x, {}^1\!/_2) = h_1(x, 0) = h_2(x, 1) = g(x).$$

Mapping $h(x, t)$ is well-defined and since both h_1 and h_2 are continuous, so is $h(x, t)$.

Thus, $h(x, t)$ is a homotopy between f and k.

The relation $\sim$ is an equivalence relation on Y^X.

• PROBLEM 19-6

1. Why do the homotopy classes of Y^X form a partition of Y^X?

2. Can you find any relationship between the extension of functions and functions being homotopic?

SOLUTION:

DEFINITION OF A HOMOTOPY CLASS

The family of all continuous functions from X into Y, which are homotopic to a continuous function f, is called the homotopy class of f.

■

The homotopy classes of Y^X form a partition of Y^X because $\sim$ is an equivalence relation on Y^X (See Problem 19–5).

2. Suppose two continuous functions f and g are given, $f, g \in Y^X$. We define function h as follows:

$$h : (X \times \{0\}) \cup (X \times \{1\}) \to Y$$

$$h(x, 0) = f(x) \quad \text{and} \quad h(x, 1) = g(x)$$

for all $x \in X$.

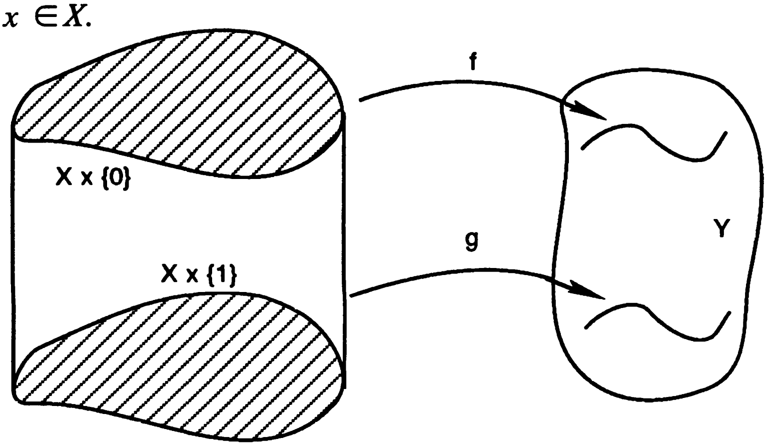


FIGURE 1

It is easy to see that f and g are homotopic, if and only if function h can be extended to a continuous function H

$$H : X \times [0, 1] \to Y$$

such that

$$H \mid (X \times \{0\}) \cup (X \times \{1\}) = h.$$

• PROBLEM 19–7

Why are any two paths in R^2 homotopic?

SOLUTION:

A continuous function from the interval $[0, 1]$ into any space (X, T) is called a path in X. (See Figure 1.)

The space

$$[0, 1] \times [0, 1]$$

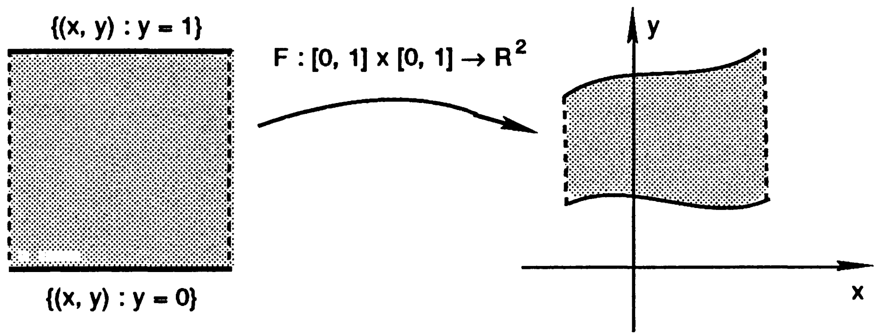
with the product topology is a normal space and its subset

$$A = \{(x, y) : y = 1\} \cup \{(x, y) : y = 0\}$$

is a closed set.

If f is any continuous function from A into R^2, then, by Tietze's Extension Theorem, f has a continuous extension F.

Thus, any two paths in R^2 are homotopic.



• PROBLEM 19-8

In Problem 19–2, we have shown that the identity function $f_1 = i$ on the unit disk Y is homotopic to the function f_0, which maps the whole disk into (0, 0). Show that $f_1 = i$ is homotopic to the function

$$g : Y \to Y$$

defined by

$$g(x, y) = (0, 1/2)$$

for all $(x, y) \in Y$.

SOLUTION:

Let h denote the homotopy between $f_1 = i$ and f_0. We define

$$H((x,y),t) = \begin{cases} h((x,y),1-2t) & \text{for } 0 \le t \le \frac{1}{2} \\ (0,t-\frac{1}{2}) & \text{for } \frac{1}{2} \le t \le 1 \end{cases}.$$

Note, that $H((x, y), t)$ on the segment $[0, 1/2]$ contracts Y to (0, 0) and then on the segment $[1/2, 1]$, slides it from (0, 0) to $(0, 1/2)$ along the straight line between these two points.

Since $H \mid (Y \times [0, {}^1\!/_2])$ and $H \mid (Y \times [{}^1\!/_2, 1])$ are continuous and

$$H((x, y), {}^1\!/_2) = h((x, y), 0) = (0, 0)$$

is well-defined, H is continuous. Thus H is a homotopy, since

$$H \mid (Y \times \{0\}) = g$$

$$H \mid (Y \times \{1\}) = f_1 = i.$$

● PROBLEM 19-9

Show, that if the homotopic mappings f_0 and f_1 have values in the space Y and if g_0 and g_1 are defined on Y and homotopic, then the mappings $g_0 \circ f_0$ and $g_1 \circ f_1$ are homotopic, i.e., prove that

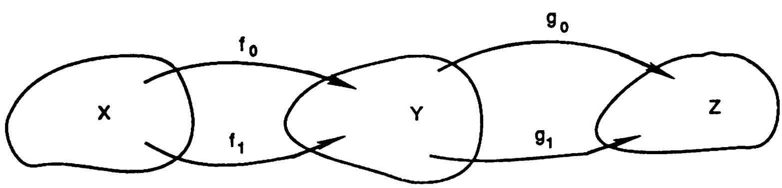


FIGURE 1

$$\begin{pmatrix} f_0 \sim f_1 \\ \\ g_0 \sim g_1 \end{pmatrix} \Rightarrow (g_0 \circ f_0 \sim g_1 \circ f_1).$$

SOLUTION:

Since $f_0 \sim f_1$, a continuous function

$$h_1(x, t) \in Y$$

exists, such that

$$h_1(x, 0) = f_0(x)\,;\; h_1(x, 1) = f_1(x).$$

Since $g_0 \sim g_1$, a continuous function

$$h_2(y, t) \in Z$$

exists, such that

$$h_2(y, 0) = g_0(y)\,;\; h_2(y, 1) = g_1(y).$$

To prove that

$$g_0 \circ f_0 \sim g_1 \circ f_1$$

we have to show that a continuous function

$$H(x, t) \in Z$$

exists, such that

$$H(x, 0) = g_0 \circ f_0 \quad \text{and} \quad H(x, 1) = g_1 \circ f_1.$$

Let us define

$$H(x, t) = h_2(h_1(x, t), t)$$

then

$$H(x, 0) = h_2(h_1(x, 0), 0) = h_2(f_0(x), 0) =$$

$$= g_0(f_0(x)) = g_0 \circ f_0(x)$$

and

$$H(x, 1) = h_2(h_1(x, 1), 1) = h_2(f_1(x), 1) =$$

$$= g_1(f_1(x)) = g_1 \circ f_1(x).$$

Since h_1 and h_2 are continuous, H is also continuous. Thus, H is a homotopy between $g_0 \circ f_0$ and $g_1 \circ f_1$.

● PROBLEM 19–10

A homeomorphism of [0, 1] into any space (X, T) is called an arc. A continuous function of [0, 1] into any space is called a path. In Problem 19–7, we showed that any two paths in R^2 are homotopic. Thus, we can state that any two arcs in R^2 are homotopic. Let A represent any point in R^2. Show that two arcs in $R^2 - \{A\}$ are not necessarily homotopic.

SOLUTION:

Let a_1 and a_2 denote distinct arcs in $R^2 - \{A\}$, such that

$$a_1(0) = a_2(0) \quad \text{and} \quad a_1(1) = a_2(1).$$

Obviously a_1 and a_2 are homotopic in R^2. We choose a_1 and a_2 in such a way, that A is a point in the area bounded by a_1 and a_2. (See Figure 1.)

Then a_1 and a_2 are not homotopic in $R^2 - \{A\}$ because a_1 cannot be continuously transformed into a_2.

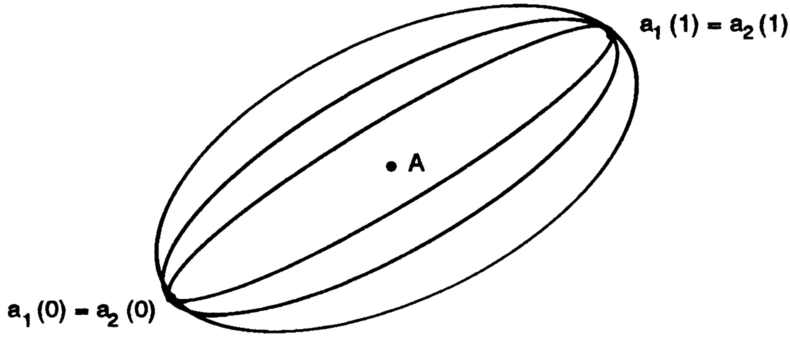


FIGURE 1

● PROBLEM 19–11

When are two loops a_0 and a_1 in X with base point x_0 homotopic relative to x_0?

SOLUTION:

A continuous function a

$$a : [0, 1] \rightarrow X$$

such that

$$a(0) = a(1) = x_0$$

is called a loop in X with base point y_0.

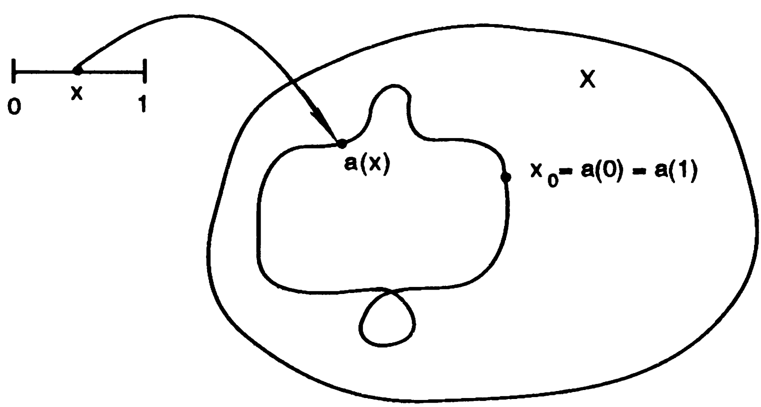


FIGURE 1

Two loops a_0 and a_1 in X with base point x_0 are said to be homotopic

relative to x_0, if a homotopy H between a_0 and a_1 exists, i.e.,

$$H(x, t) \in X;\ x, t \in [0, 1]$$

$$H(x, 0) = a_0\ ;\ H(x, 1) = a_1$$

such that, for each $t \in [0, 1]$

$$H(0, t) = H(1, t) = x_0.$$

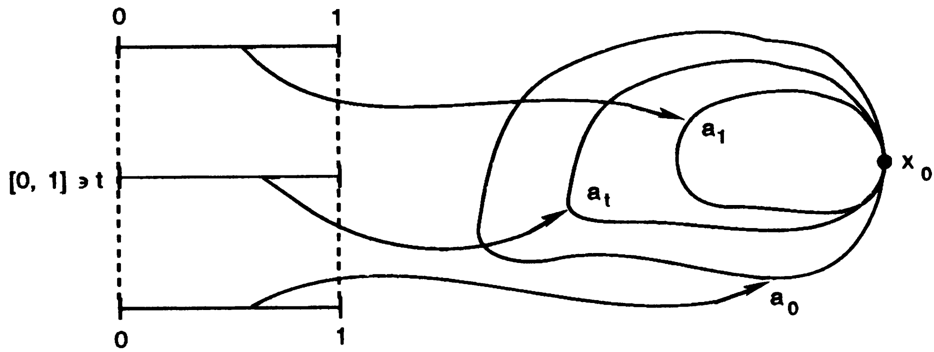


FIGURE 2

$$a_t(0) = a_t(1) = x_0$$

That is, for each $t \in [0, 1]$

$$H \mid [0, 1] \times \{t\}$$

is a loop in X with base point x_0. See Figure 2.

● PROBLEM 19–12

Let $L(X, x_0)$ denote the set of all loops in X with base point x_0. Show that the relation "homotopic relative to x_0 to" (denoted by ~) defined on $L(X, x_0)$, is an equivalence relation.

SOLUTION:

In Problem 19–5, we showed that "f, homotopic to g", is an equivalence relation on Y^X.

For any

$$a \in L(X, x_0)$$

a is homotopic relative to x_0 to a

$$a \sim a.$$

Let

$$a \sim b.$$

Then, the homotopy H exists between a and b, such that

$$H(0, t) = H(1, t) = x_0.$$

Setting

$$H'(x, t) = H(x, 1 - t)$$

we obtain homotopy between b and a, such that

$$H'(0, t) = H'(1, t) = x_0.$$

Hence,

$$b \sim a.$$

Suppose $a \sim b$ and $b \sim c$.

Homotopies exist, such that

$$h_1(0, t) = h_1(1, t) = x_0$$

$$h_1(0, t) = h_2(1, t) = x_0.$$

Then

$$h(x,t) = \begin{cases} h_2(x, 2t) & \text{for} \quad 0 \le t \le \frac{1}{2} \\ h_1(x, 2t-1) & \text{for} \quad \frac{1}{2} \le t \le 1 \end{cases}$$

is a homotopy between a and c, such that

$$h(0, t) = h(1, t) = x_0.$$

● PROBLEM 19-13

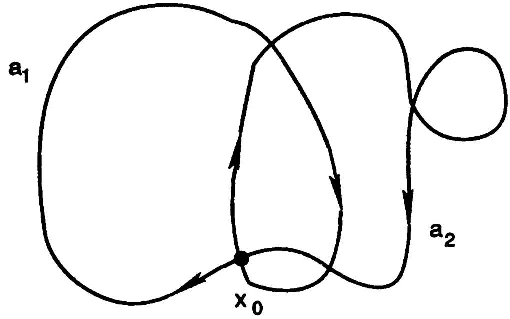
Let $L(X, x_0)$ denote the set of all loops in X with base point x_0. For any a_1, $a_2 \in L(X, x_0)$, we define an operation $\oplus$ as follows:

$$(a_1 \oplus a_2)(t) = \begin{cases} a_1(2t), & 0 \le t \le \frac{1}{2} \\ a_2(2t-1), & \frac{1}{2} \le t \le 1 \end{cases}.$$

Show that

$$a_1 \oplus a_2 \in L(X, x_0).$$

See Figure 1.



$a_1 \oplus a_2$

FIGURE 1

SOLUTION:

Loop $a_1 \oplus a_2$ is formed by moving arround a_1 once and then around a_2 once.

$$a_1 \oplus a_2 : [0, 1] \to X.$$

$a_1 \oplus a_2$ is continuous on $[0, {}^1\!/_2]$ and on $[{}^1\!/_2, 1]$ because both a_1 and a_2 are continuous and at $t = {}^1\!/_2$

$$(a_1 \oplus a_2)\,({}^1\!/_2) = a_1(1) = x_0 = a_2(0).$$

Also

$$(a_1 \oplus a_2)\,(0) = a_1(0) = x_0$$

$$(a_1 \oplus a_2)\,(1) = a_2(1) = x_0.$$

Hence,

$$a_1 \oplus a_2 \in L(X, x_0).$$

• PROBLEM 19–14

We proved that ~ (is homotopic relative y_0 to) is an equivalence relation on $L(X, x_0)$. We denote

$$\frac{L(X,x_0)}{\sim} = \pi(X,x_0)$$

the set of homotopy relative to x_0 classes of $L(X,x_0)$. If $a \in L(X, x_0)$, then by

$$[a] \in \pi(X, x_0)$$

we denote the equivalence class of a.

Justify the following definition:

$$[a_1] \oplus [a_2] = [a_1 \oplus a_2].$$

SOLUTION:

We define $[a_1] \oplus [a_2]$ to be the equivalence class of the loop $a_1 \oplus a_2$. In Problem 19–13, we proved that $a_1 \oplus a_2$ is a loop.

$$a_1 \oplus a_2 \in L(X, x_0).$$

Taking its homotopy equivalence class $[a_1 \oplus a_2]$, we obtain an element of $\pi(X, x_0)$.

To make sure that $[a_1] \oplus [a_2] = [a_1 \oplus a_2]$ is a well-defined operation on $\pi(X, x_0)$, it must be shown that this definition is independent of the representatives a_1 and a_2 that we choose. This will be done in the next problem.

● PROBLEM 19–15

Let

$$a_1, a_2, a_3 \in L(X, x_0).$$

Show, that if $a_1 \sim a_3$, then

$$a_1 \oplus a_2 \sim a_3 \oplus a_2.$$

SOLUTION:

Since $a_1 \sim a_3$, a homotopy relative to x_0 exists between a_1 and a_3, such that

$$h : [0, 1] \times [0, 1] \to X.$$

We define

$$H : [0, 1] \times [0, 1] \to X$$

by

$$H(r,t) = \begin{cases} h(2r,t), & 0 \le r \le \frac{1}{2} \\ a_2(2r-1), & \frac{1}{2} \le r \le 1 \end{cases}$$

Thus,

$$H(r,0) = \begin{cases} h(2r,0) = a_1(2r), & 0 \le r \le \frac{1}{2} \\ a_2(2r-1), & \frac{1}{2} \le r \le 1 \end{cases}$$

and

$$H(r,\ 0) = (a_1 \oplus a_2)\ (r) \text{ for } r \in [0,\ 1].$$

Also

$$H(r,\ 1) = (a_3 \oplus a_2)\ (r) \text{ for } r \in [0,\ 1].$$

We have

$$H(0,\ t) = h(0,\ t) = x_0$$

$$H(1,\ t) = \mathrm{a}_2(1) = x_0.$$

Continuity of H can easily be shown. Hence,

$$a_1 \oplus a_2 \sim a_3 \oplus a_2.$$

● PROBLEM 19–16

Show that the operation

$$[a_1] \oplus [a_2] = [a_1 \oplus a_2] \tag{1}$$

defined in Problem 19–14 is well-defined.

SOLUTION:

Suppose $a_1 \sim a_3$ and $a_2 \sim a_4$. In Problem 19–15, we proved that

$$a_1 \oplus a_2 \sim a_3 \oplus a_2. \tag{2}$$

Similary, we can show that

$$a_1 \oplus a_2 \sim a_1 \oplus a_4. \tag{3}$$

Thus,

$$[a_1 \oplus a_2] = [a_1] \oplus [a_2] = [a_3 \oplus a_2] =$$

$$= [a_3] \oplus [a_2] \tag{4}$$

and

$$[a_1] \oplus [a_2] = [a_1] \oplus [a_4]. \tag{5}$$

We have

$$a_1 \oplus a_2 \sim a_3 \oplus a_2 \sim a_3 \oplus a_4. \tag{6}$$

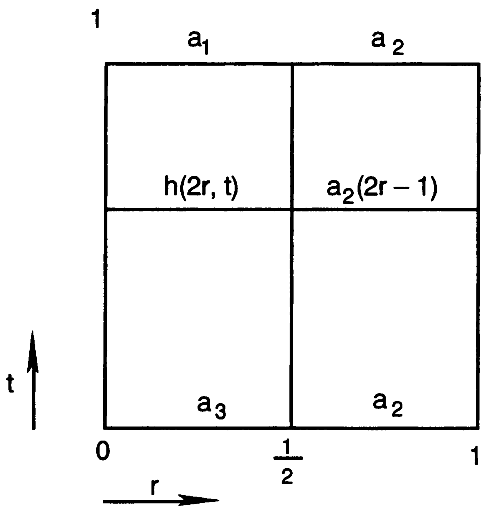
Hence, if $[a_1] = [a_3]$ and $[a_2] = [a_4]$, then

$$[a_1] \oplus [a_2] = [a_3] \oplus [a_4]. \tag{7}$$

• PROBLEM 19–17

> Draw the homotopy $H(r, t)$ described in Problem 19–15. Explain why $H(r, t)$ is continuous on $[0, 1] \times [0, 1]$.

SOLUTION:



H(r, t)

FIGURE 1

On $[0, {}^1/_2] \times [0, 1]$, we continuously deform a_1 into a_3, while a_2 remains fixed on $[{}^1/_2, 1] \times [0, 1]$.

For $r = {}^1/_2$, we have

$$H({}^1/_2, t) = h(1, t) = a_2(1) = x_0$$

$$H({}^1/_2, t) = a_2(0) = x_0.$$

The homotopy $H(r, t)$ is continuous on $[0, {}^1/_2] \times [0, 1]$ and on $[{}^1/_2, 1] \times [0, 1]$, hence, it is continuous on $[0, 1] \times [0, 1]$.

● PROBLEM 19–18

Show that operation $\oplus$, defined on $\pi(X, x_0)$, is an associative operation; that is, show that

$$([a_1] \oplus [a_2]) \oplus [a_3] = [a_1] + ([a_2] \oplus [a_3]).$$

SOLUTION:

Let $[a_1]$, $[a_2]$, $[a_3]$ represent any elements of $\pi(X, x_0)$. Since operation $\oplus$ defined on $\pi(X, x_0)$ does not depend on the choice of representatives, it suffices to show, that

$$(a_1 \oplus a_2) \oplus a_3 \sim a_1 \oplus (a_2 \oplus a_3).$$

By definition

$$((a_1 \oplus a_2) \oplus a_3)(r) = \begin{cases} (a_1 \oplus a_2)(2r); & 0 \le r \le \frac{1}{2} \\ a_3(2r-1); & \frac{1}{2} \le r \le 1 \end{cases}.$$

Thus,

$$((a_1 \oplus a_2) \oplus a_3)(r) = \begin{cases} a_1(4r); \; 0 \le r \le \frac{1}{4} \\ a_2(4r-1); & \frac{1}{4} \le r \le \frac{1}{2}. \\ a_3(2r-1); & \frac{1}{2} \le r \le 1 \end{cases}$$

Similarly,

$$(a_1 \oplus (a_2 \oplus a_3)(r) = \begin{cases} a_1(2r); & 0 \le r \le \frac{1}{2} \\ (a_2 \oplus a_3)(2r-1); & \frac{1}{2} \le r \le 1 \end{cases} =$$

$$= \begin{cases} a_1(2r); \; 0 \le r \le \frac{1}{2} \\ a_2(4r-2); & \frac{1}{2} \le r \le \frac{3}{4}. \\ a_3(4r-3); & \frac{3}{4} \le r \le 1 \end{cases}$$

The homotopy between $(a_1 \oplus a_2) \oplus a_3$ and $a_1 \oplus (a_2 \oplus a_3)$ is defined as follows:

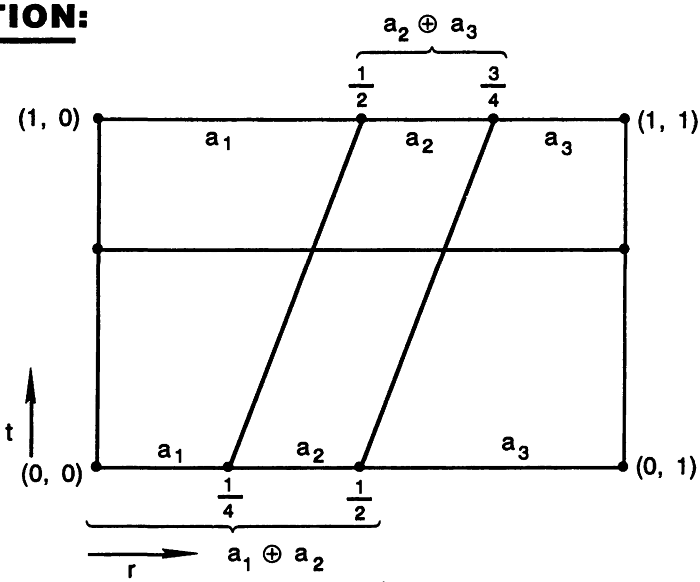
$$H(r,t)=\begin{cases} a_1\left(\frac{4r}{1+t}\right); & 0\le r\le \frac{1}{4}(1+t) \\ a_2(4r-1-t); & \frac{1}{4}(1+t)\le r\le \frac{1}{4}(2+t). \\ a_3\left(1-\frac{4(1-r)}{2-t}\right); & \frac{1}{4}(2+t)\le r\le 1 \end{cases}$$

One can directly compute that $H(r, t)$ is indeed the homotopy between $(a_1 \oplus a_2) \oplus a_3$ and $a_1 \oplus (a_2 \oplus a_3)$.

• PROBLEM 19–19

Draw the homotopy defined in Problem 19–18. Explain why it is a suitable homotopy.

SOLUTION:



H(r, t)

FIGURE 1

The homotopy $H(r, t)$ was defined as follows:

$$H(r,t)=\begin{cases} a_1\left(\frac{4r}{1+t}\right); & 0\le r\le \frac{1}{4}(1+t) \\ a_2(4r-1-t); & \frac{1}{4}(1+t)\le r\le \frac{1}{4}(2+t). \\ a_3\left(1-\frac{4(1-r)}{2-t}\right); & \frac{1}{4}(2+t)\le r\le 1 \end{cases}$$

Note that for $t = 0$

$$H(r,0) = \begin{cases} a_1(4r); & 0 \le r \le \frac{1}{4} \\ a_2(4r-1); & \frac{1}{4} \le r \le \frac{1}{2} \\ a_3(2r-1); & \frac{1}{2} \le r \le 1 \end{cases}$$

which is $((a_1 \oplus a_2) \oplus a_3)(r)$.

Similarly, for $t = 1$, we have

$$H(r,1) = \begin{cases} a_1(2r); & 0 \le r \le \frac{1}{2} \\ a_2(4r-2); & \frac{1}{2} \le r \le \frac{3}{4} \\ a_3(4r-3); & \frac{3}{4} \le r \le 1 \end{cases}$$

which is $(a_1 \oplus (a_2 \oplus a_3))(r)$. It is evident that $H(r, t)$ is continuous.

● PROBLEM 19-20

Let

$$f: [0, 1] \to x_0 \in X$$

denote a constant function that maps [0, 1] onto $x_0 \in X$. Show that $[f]$ is an identity for $\pi(X, x_0)$ with respect to $\oplus$.

SOLUTION:

We shall prove, that for any $[a] \in \pi(X, x_0)$,

$$[a] \oplus [f] = [f] \oplus [a] = [a].$$

By definition

$$[a] \oplus [f] = [a \oplus f]$$

and

$$(a \oplus f)(r) = \begin{cases} a(2r); & 0 \le r \le \frac{1}{2} \\ f(2r-1); & \frac{1}{2} \le r \le 1 \end{cases}.$$

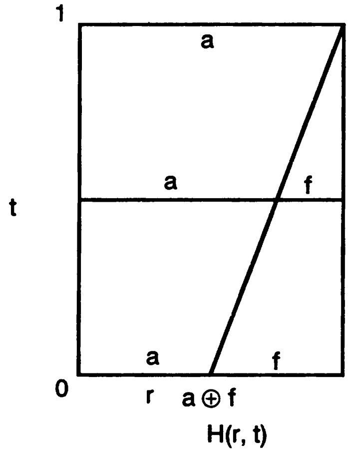


FIGURE 1

The figure illustrates the homotopy between $a \oplus f$ and a. Hence,

$$[a] \oplus [f] = [a].$$

Similarly, we show that

$$[f] \oplus [a] = [a].$$

● PROBLEM 19–21

Let a denote a loop in X with base point x_0, $a \in L(X, x_0)$. Define an inverse loop a^{-1}.

SOLUTION:

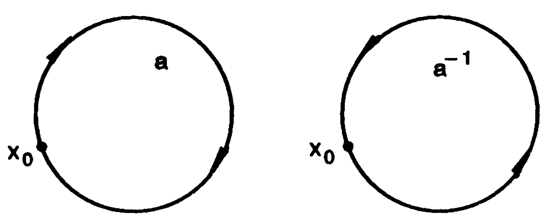


FIGURE 1

We define a^{-1} by

$$a^{-1}(r) = a(1 - r)$$

for each $r \in [0, 1]$.

Then

$$a^{-1}(0) = a(1) = x_0$$

and

$$a^{-1}(1) = a(0) = x_0.$$

Also, since a is continuous, so is a^{-1}, hence,

$$a^{-1} \in L(X, x_0).$$

Note that a^{-1} has the shape of a, but is drawn in the opposite direction.

• PROBLEM 19–22

Define the fundamental group of the space X, given that $x_0 \in X$.

SOLUTION:

A topological space (X, T) is given and $x_0 \in X$. $L(X, x_0)$ is the set of all loops in X with base point x_0.

On the set of homotopy, the classes of $L(X, x_0)$ denoted by $\pi(X, x_0)$, we defined operation $\oplus$, which is associative and has an identity $[f]$ with respect to x.

It is easy to prove, that

$$a \oplus a^{-1} \sim a^{-1} \oplus a \sim f$$

by using the homotopy between $a \oplus a^{-1}$ and f

$$H(r,t) = \begin{cases} a(2r(1-t)); & 0 \le r \le \frac{1}{2} \\ a(2(1-r)(1-t)); & \frac{1}{2} \le r \le 1 \end{cases}.$$

Similarly, we can define the homotopy between $a^{-1} \oplus a$ and f. Thus, the inverse of $[a] \in \pi(X, x_0)$ is $[a^{-1}]$.

The space $\pi(X, x_0)$ with operation $\oplus$ is a group with identity $[f]$. Each $[a] \in \pi(X, x_0)$ has its inverse $[a^{-1}] \in \pi(X, x_0)$.

DEFINITION

$(\pi(X, x_0), \oplus)$ is called the fundamental group of the space X based on $x_0 \in X$.

■

Consider the space

$$X = \{(x, y) : x^2 + y^2 = 1\} \subset R^{2.}$$

Let x_0 represent any point of X and a the loop $a \subset X$, which goes once around the circle in the clockwise direction. For each integer $n \in Z$ and each loop $a \in L(X, x_0)$, we define

$$na = \begin{cases} \overbrace{a \oplus a \oplus \ldots \oplus a}^{n \text{ times}} & \text{for } n \text{ positive} \\ f & \text{for } n = 0 \\ \underbrace{a^{-1} \oplus a^{-1} \oplus \ldots a^{-1}}_{-n \text{ times}} & \text{for } n \text{ negative} \end{cases}$$

where

$$f : [0, 1] \to x_0.$$

Find the fundamental group of X based on x_0.

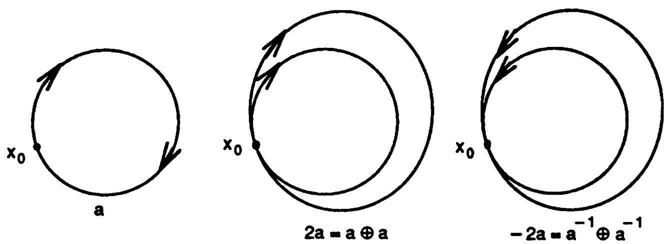


FIGURE 1

SOLUTION:

Note, that

$$m\,a \sim n\,a$$

if and only if $m = n$. For every loop $a_1 \in L(X, x_0)$, there is $n \in Z$, such that

$$a_1 \in [n\,a].$$

Thus, there is an isomorphism h between

$$(\pi(X, x_0), \oplus) \text{ and } (Z, +)$$

given by

$$h([n\,a]) = n.$$

Group $(\pi(X, x_0), \oplus)$ is generated by one element $[a]$.

Find the fundamental group based on x_0 of the space X

$$X = \{(x, y, z) : x^2 + y^2 + z^2 = 1\} \subset R^3$$

$x_0 \in X$.

SOLUTION:

Let A represent any point of X. Then $X - \{A\}$ is homeomorphic to R^2 and therefore, a contractible space. The plane λ is tangent to X at A'. A' is the point of X antipodal to A.

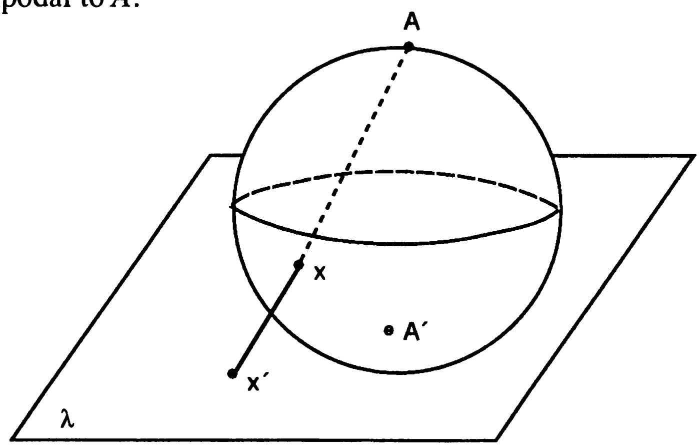


FIGURE 1

The projection of a point $x \in X - \{A\}$ is a point $x' \in \lambda$ determined by the intersection of a straight line $\overline{Ax}$ with the plane λ. Thus, $X - \{A\}$ and λ are homeomorphic.

If a is any loop in X and A is any point in $X - a$, then a is a loop in R^2 (remember R^2 and $X - \{A\}$ are homeomorphic).

Thus, a is homotopic to the identity function f of $\pi(X, x_0)$. Hence, any loop in X based on $x_0 \in X$ is homotopic to f.

Therefore, $\pi(X, x_0)$ consists only of $[f]$.

GLOSSARY OF SYMBOLS AND ABBREVIATIONS

Symbol	Meaning	See Problem where symbol appears
$\alpha, \beta, \gamma, \delta, \ldots$	sentences	1-1
0,1	logical values	1-1
$\equiv$	equivalent	1-1
$\vee$	or	1-1
$\wedge$	and	1-1
$'$	not	1-1
$\Rightarrow$	implies	1-5
$\Leftrightarrow$	if and only if	1-5
iff	if and only if	1-5
$\dotplus$	symmetric difference	1-11
$\blacksquare$	end of a proof, theorem, or definition	1-6
$a \in A$	a is an element of A	2-1
$a \notin A$	a is not an element of A	2-1
$\{a, b, c, \ldots\}$	set consisting of elements $a, b, c, \ldots$	2-1
$\{x : p(x)\}$	set of all x such that $p(x) \equiv 1$	2-1
N	set of all natural numbers	
Z	set of all integers	
Q	set of all rational numbers	
R	set of all real numbers	
C	set of all imaginary numbers	
$A \subset B$	A is a subset of B	2-3
$A \cup B$	union of A and B	2-4
$A \cap B$	intersection of A and B	2-4
$A - B$	difference of A and B	2-4
ϕ	empty set	2-7
$P(A)$ (or 2^A)	power set of A	2-8
A^C (or $X - A$)	complement of A	2-12

$A \div B$	symmetric difference of A and B	2-18
$\exists$	there exists	3-2
$\forall$	for each	3-2
(x_n)	sequence consisting of elements $x_1, x_2, \ldots$	3-4
$\lim\limits_{n\to\infty} x_n$	limit of (x_n)	3-4
(a, b)	ordered pair	3-5
$A \times B$	Cartesian product	3-5
$]a, b[$ (or (a, b))	open interval	3-9
$\bigcup\limits_{\alpha} A_\alpha$	union of a family of sets	3-10
$\bigcap\limits_{\alpha} A_\alpha$	intersection of a family of sets	3-10
$[a, b]$	closed interval	3-11
$a \rho b$	a is in relation ρ with b	3-13
ρ^{-1}	inverse relation	3-14
Δ	diagonal of a set	3-14
$\rho \circ \lambda$	composition of relations	3-15
R	equivalence relation	3-17
$\frac{X}{R}$	quotient space	3-23
$f : X \to Y$	function defined on X	4-1
$f \mid A$	restriction of f to A	4-3
$f \circ g$	composition of functions	4-5
1_A	identity function on A	4-8
A^B	set of all functions $f : B \to A$	4-22
χ_A	characteristic function of A	4-25
$\underset{\alpha}{X} A_\alpha$ (or $\prod\limits_{\alpha} A_\alpha$)	Cartesian product of a family of sets	4-28
$A \sim B$	sets A and B are equivalent	5-1
card$A$ (or $\overline{\overline{A}}$)	cardinal number of set A	5-4
$\alpha + \beta$	sum of cardinal numbers	6-14
$\alpha \cdot \beta$	product of cardinal numbers	6-16

β^α	α^{th} power of β	6-23
H	Hilbert space	6-37
$(A, <)$	preordered set	7-4
$(A, \leq)$	partially ordered set	7-7
$=:$	equals by definition	7-10
$(A, \leq)$	totally ordered set	7-8
lub	the least upper bound	7-18
glb	the greatest lower bound	7-18
$A \approx B$	order – isomorphic sets	8-2
ord $(A, \leq)$	ordinal number of a well-ordered set A	8-4
$d(x, y)$	metric	10-1
(X, d)	metric space	10-1
R^n	n-dimensional Euclidean space	10-3
$B_d(a, r)$	d-ball of radius r and center a	10-7
$\overline{B(a, r)}$	closed ball	10-14
(x_n)	sequence of points	10-17
$\lim_{n \to \infty} x_n$	limit of a sequence	10-17
$\delta(A)$	diameter of A	10-26
$\overline{A}$	closure of A	10-32
(X, T)	topological space	11-1
$T(A)$	topology generated by A	11-18
$N_0(x)$	neighborhood of x	11-27
$\overline{A}$	closure of A	11-27
A^1	derived set of A	11-31
Int(A)	interior of A	11-34
Fr(A)	boundary of A	11-35
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